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The well-posedness of stochastic Kawahara equation: fixed point argument and Fourier restriction method

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Abstract

In this paper, we investigate the Cauchy problem for the stochastic Kawahara equation, which is a fifth-order shallow water wave equation. We prove local well-posedness for data in $H^s(\mathbb{R})$, $s > -7/4$. Moreover, we get global existence for $L^2(\mathbb{R})$ solutions. Due to the non-zero singularity of the phase function, a fixed point argument and Fourier restriction method are proposed.

Keywords: Kawahara equation, Well-posedness, Wiener process, Fixed point theorem, Fourier restriction method

AMS Subject Classification: 60H15, 49K40, 60H40

Introduction

In this paper, we consider the Cauchy problem for the stochastic Kawahara equation:

$$u_t + \alpha u_{5x} + \beta u_{3x} + \gamma u_x + \mu u u_x = \Phi \frac{\partial^2 B}{\partial t \partial x}, \tag{1}$$

where $\alpha \neq 0$, β, and γ are real numbers; μ is a complex number; u is a stochastic process defined on $(x,t) \in \mathbb{R} \times \mathbb{R}_+$; Φ is a linear operator; and B is a two-parameter Brownian motion on $\mathbb{R} \times \mathbb{R}_+$, that is, a zero mean Gaussian process whose correlation function is given by:

$$\mathbb{E}\left(B(x,t)B(y,s)\right) = (x \wedge y)(t \wedge s), \quad t,s \geq 0,\ x,y \in \mathbb{R}. \tag{2}$$

In general, the covariance operator Φ can be described by a kernel $\mathcal{K}(x,y)$. The correlation function of the noise is then given by

$$\mathbb{E}\left(\Phi \frac{\partial^2 B}{\partial t \partial x}(x,t) \Phi \frac{\partial^2 B}{\partial t \partial x}(y,s)\right) = c(x,y)\delta_{t-s},$$

where $t,s \geq 0$, $x,y \in \mathbb{R}$, δ is the Dirac function and

$$c(x,y) = \int_{\mathbb{R}} \mathcal{K}(x,z)\mathcal{K}(y,z)dz.$$

Consider a fixed probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$. As usual, we can rewrite the right hand side of Eq. (1) as the time derivative of a cylindrical Wiener process on $L^2(\mathbb{R})$ by setting:

$$W(t) = \frac{\partial B}{\partial x} = \sum_{i\in\mathbb{N}} \beta_i(t) e_i, \tag{3}$$

where $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R})$ and $(\beta_i)_{i\in\mathbb{N}}$ is a sequence of mutually independent real Brownian motions in $(\Omega, \mathcal{F}, P)$. Let us rewrite Eq. (1) in its Itô form as follows:

$$\begin{cases} du + (\alpha u_{5x} + \beta u_{3x} + \gamma u_x + \mu u u_x)\, dt = \Phi dW(t), \\ u(x,0) = u_0(x) \end{cases} \tag{4}$$

In order to obtain local well-posedness of Eq. (1), we mainly work on the general mild formulation of Cauchy problem (4) as below:

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\,(\mu u u_x)\, ds + \int_0^t U(t-s)\Phi dW(s). \tag{5}$$

Here, $U(t) = \mathfrak{F}_x^{-1} \exp\left(-it\phi(\xi)\right) \mathfrak{F}_x$ is the unitary group of operators related to the linearized equation:

$$u_t + \alpha u_{5x} + \beta u_{3x} + \gamma u_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{6}$$

where $\phi(\xi) = \alpha\xi^5 - \beta\xi^3 + \gamma\xi$ is the phase function and $\mathfrak{F}_x$ (or " $\hat{\cdot}$ ") is the usual Fourier transform in the x variable. We note that the phase function ϕ has non-zero singularity. This differs from the phase function of the linear Korteweg-de Vries (KdV) equation (see [1]) and causes some difficulties in the problem. To avoid these difficulties, we eliminate the singularity of the phase function ϕ by using the Fourier restriction operators [2]:

$$P^N f = \int_{|\xi|\geq N} e^{ix\xi}\hat{f}(\xi) d\xi, \qquad P_N f = \int_{|\xi|\leq N} e^{ix\xi}\hat{f}(\xi) d\xi, \;\; \forall N > 0.$$

In the case of $\Phi \equiv 0$ (effect of the noise does not exist), Eq. (1) is reduced to the deterministic Kawahara equation:

$$u_t + \alpha u_{5x} + \beta u_{3x} + \gamma u_x + \mu u u_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{7}$$

As aforesaid in [3–5], Eq. (7) is a fifth-order shallow water wave equation. It arises in study of the water waves with surface tension, in which the Bond number takes on the critical value, where the Bond number represents a dimensionless magnitude of surface tension in the shallow water regime. If we consider a realistic situation, in which a non-constant pressure affects on the surface of the fluid or the bottom of the layer is not flat, it is meaningful to add a forcing term to Eq. (7). This term can be given by the gradient of the exterior pressure or of the function whose graph defines the bottom [6, 7]. This paper focuses on the case when the forcing term is of additive white noise type. This leads us to study the stochastic fifth-order shallow water wave Eq. (1). By means of white noise functional analysis, the analytical white noise functional solutions for the nonlinear stochastic partial differential equations (SPDEs) can be investigated. This subject is attracting more and more attention [8–15].

It is well known that the Cauchy problem (4) is locally well-posed for data in $H^s(\mathbb{R})$, $s \in \mathbb{R}$, if for any finite time T, there exists a locally continuous mapping that transfers $u_0 \in$

$H^s(\mathbb{R})$ to a unique solution $u \in C([0,T];H^s(\mathbb{R}))$. If the solution mapping exists for all time, we say that the Cauchy problem (4) is globally well-posed [16].

In [17], Huo obtained a local well-posedness result in $H^s(\mathbb{R})(s > -11/8)$ for the Kawahara equation. Moreover, Jia and Huo [18] proved the local well-posedness of the Kawahara and modified Kawahara equations for data in $H^s(\mathbb{R})$ with $s > -7/4$ and $s \geq -1/4$ respectively. The first well-posedness result for the Kaup-Kupershmidt equations was presented by Tao and Cui [19]. They proved that their Cauchy problems are locally well-posed in $H^s(\mathbb{R})$ for $s > 5/4$ and $s > 301/108$, respectively. Thereafter, Zhao and Gu [20] lowered the regularity of the initial data space to $s > 9/8$ and improved the preceding result in [19]. Also, using a Fourier restriction method, a local well-posedness result for the Kaup-Kupershmidt equations was established in [18] for data in $H^s(\mathbb{R})$ with $s > 0$ and $s > -1/4$, respectively.

If $\alpha = \gamma = 0$, the model (7) is minified to the famous KdV equation:

$$u_t + \beta u_{3x} + \mu u u_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{8}$$

The well-posedness of Eq. (8) was studied by Kenig, Ponce, and Vega [21]. They proved that its Cauchy problem is locally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$. Also, Ponce [1] discussed the general fifth-order shallow water wave equation:

$$u_t + u_x + c_1 u u_x + c_2 u_{3x} + c_3 u_x u_{xx} + c_4 u u_{3x} + c_5 u_{5x} = 0 \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{9}$$

and gave a global well-posedness result of its Cauchy problem for data in $H^4(\mathbb{R})$. The well-posedness of the SPDEs has been the subject of a large amount of work. de Bouard and Debussche [22] considered the stochastic KdV equation forced by a random term of white noise type. They proved existence and uniqueness of solutions in $H^1(\mathbb{R})$ and existence of martingale solutions in $L^2(\mathbb{R})$ in the case of additive and multiplicative noise, respectively. Since that time, many researchers paid more attention to investigate the Cauchy problems for some SPDEs and have obtained a number of local and global well-posedness results [23–25].

The goal of this paper is to investigate the Cauchy problem of the stochastic Kawahara Eq. (1), where the random force is of additive white noise type. By employing a Fourier restriction method, a Banach fixed point theorem, and some basic inequalities, we show that Eq. (1) is locally well-posed for data in $H^s(\mathbb{R})$, $s > -7/4$. Also, we give global existence for $L^2(\mathbb{R})$ solutions. An outline of this paper is as follows. The "Main results" section contains precise statement of our new results and some important function spaces. In the section "The stochastic convolution estimate", we give an estimation of the stochastic convolution term via a Fourier restriction method and some basic inequalities. In the section "Local well-posedness: proof of Theorem 1", we use the stochastic estimation proved in the section "The stochastic convolution estimate" and the Banach fixed point theorem to obtain a local well-posedness result of Eq. (1). In the section "Global well-posedness: proof of Theorem 2", we extend our technique and show global well-posedness result of Eq. (1). The "Summary and discussion" section is devoted to the summary and discussion.

Main results

Before giving the precise statement of our main results, we introduce some notations and assumptions.

Definition 1 *For $s, b \in \mathbb{R}$ the space $\mathfrak{X}_{s,b}$ is defined to be the completion of the Schwartz function space $\mathcal{S}\left(\mathbb{R}^2\right)$ with respect to the norm:*

$$\|u\|_{\mathfrak{X}_{s,b}} = \|U(-t)u\|_{H_x^s H_t^b} = \|\langle\xi\rangle^s \langle\tau + \phi(\xi)\rangle^b \mathfrak{F}u\|_{L_\xi^2 L_\tau^2}, \tag{10}$$

where $\langle\cdot\rangle = 1 + |\cdot|$.

Definition 2 *For $T > 0$, $\mathfrak{X}_{s,b}^T$ is the space of restrictions to $[0, T]$ of functions in $\mathfrak{X}_{s,b}$ endowed with the norm:*

$$\|u\|_{\mathfrak{X}_{s,b}^T} = \inf\{\|\tilde{u}\|_{\mathfrak{X}_{s,b}} : \tilde{u} \in \mathfrak{X}_{s,b}, u = \tilde{u}|_{[0,T]}\}. \tag{11}$$

Theorem 1 *Assume that $s > -\frac{7}{4}$, $\Phi \in L_2^{0,s}$, $b \in \left(0, \frac{1}{2}\right)$ and b is close enough to $\frac{1}{2}$. If $u_0 \in H^s(\mathbb{R})$ for almost surely $\omega \in \Omega$ and u_0 is $\mathcal{F}_0-$measurable. Then for almost surely $\omega \in \Omega$, there exists a constant $T_\omega > 0$ and a unique solution u of the Cauchy problem (4) on $[0, T_\omega]$ which satisfies:*

$$u \in C\left([0, T_\omega]; H^s(\mathbb{R})\right) \cap \mathfrak{X}_{s,b}^{T_\omega}.$$

In fact the L^2-norm is preserved for a solution of the Kawahara equation [4]. Therefore, in the case of $s = 0$, we can obtain a global existence result for Eq. (1). Precisely, we have:

Theorem 2 *Let $u_0 \in L^2\left(\Omega, L^2(\mathbb{R})\right)$ be an $\mathcal{F}_0-$measurable initial data, and let $\Phi \in L_2^{0,0}$. Then, the solution u given by Theorem 1 is global and satisfies:*

$$u \in L^2\left(\Omega; C\left([0, T_0]; H^s(\mathbb{R})\right)\right), \quad \textit{for any } \; T_0 > 0.$$

The stochastic convolution estimate

In this section, using the Fourier restriction method, the properties of Itô stochastic integral and some basic inequalities, we give an estimation for the last term in Eq. (5), which is the stochastic convolution:

$$u_l(t) := \int_0^t U(t-s)\Phi dW(s). \tag{12}$$

Choose $\chi \in C_0^\infty(\mathbb{R}_+)$ such that $\chi(t) = 0$ for $t > 0$, $\chi(t) = 1$ for $0 < t < 1$ and $\chi(t) = 0$ for $t \geq 2$. Hence, $\chi \in H^b(\mathbb{R})$ for any $b < \frac{1}{2}$. Let $H_t^b := H^b([0, T]; \mathbb{R})$ be the Sobolev space in the time variable t with the norm:

$$\|\psi\|_{H_t^b}^2 := \|\psi\|_{L^2(\mathbb{R})}^2 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\psi(t_1) - \psi(t_2)|^2}{|t_1 - t_2|^{1+2b}} dt_1 dt_2, \quad \psi \in H_t^b. \tag{13}$$

Now, we state and prove the estimation of the stochastic convolution (12) as follows:

Lemma 1 *Assume that $s, b \in \mathbb{R}$ with $b \in \left(0, \frac{1}{2}\right)$, and let $\Phi \in L_2^{0,s}$ Then, u_l defined by (12) satisfies:*

$$\chi u_l \in L^2\left(\Omega, \mathfrak{X}_{s,b}\right)$$

and

$$\mathbb{E}\left(\|\chi u_l\|^2_{\mathfrak{X}_{s,b}}\right) \leq N(b,\chi)\|\Phi\|^2_{L_0^{0,s}}, \tag{14}$$

where $N(b,\chi)$ *is a constant that depends on b,* $\|\chi\|_{H_t^b}$, $\||t|^{\frac{1}{2}}\chi\|_{L_t^2}$ *and* $\||t|^{\frac{1}{2}}\chi\|_{L_t^\infty}$,

Proof Let us introduce the function

$$w(t,.) = \chi(t)\int_0^t U(-s)\Phi dW(s), \quad t \in \mathbb{R}_+. \tag{15}$$

This implies that $U(t)w(t,.) = \chi(t)u_l(t)$. Thus, by Eq. (10), we have:

$$\begin{aligned}\mathbb{E}\left(\|\chi u_l\|^2_{\mathfrak{X}_{s,b}}\right) &= \mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}(1+|\xi|)^{2s}(1+|\tau|)^{2b}|\mathfrak{F}_x w(t,\xi)|^2 d\tau d\xi\right)\\ &= \int_{\mathbb{R}}(1+|\xi|)^{2s}\,\mathbb{E}\left(\|\mathfrak{F}_x w(.,\xi)\|^2_{H_t^b}\right)d\xi,\end{aligned} \tag{16}$$

According to the expansion (3) of the cylindrical Wiener process and Eq. (13), we have:

$$\mathbb{E}\left(\|\mathfrak{F}_x w(.,\xi)\|^2_{H_t^b}\right) = S_1 + S_2, \tag{17}$$

where,

$$S_1 = \sum_{i\in\mathbb{N}}|\hat{\Phi e_i}|^2\left[\mathbb{E}\left(\left\|\chi(t)\int_0^t e^{is\phi(\xi)}d\beta_i(s)\right\|^2_{L^2(\mathbb{R})}\right)\right], \tag{18}$$

$$S_2 = \sum_{i\in\mathbb{N}}|\hat{\Phi e_i}|^2\left[\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\left|\begin{array}{c}\chi(t_1)\int_0^{t_1}e^{is\phi(\xi)}d\beta_i(s)\\ -\chi(t_2)\int_0^{t_2}e^{is\phi(\xi)}d\beta_i(s)\end{array}\right|^2}{|t_1-t_2|^{1+2b}}dt_1dt_2\right)\right]. \tag{19}$$

□

From the Itô isometry formula, we get:

$$\begin{aligned}S_1 &= \sum_{i\in\mathbb{N}}|\hat{\Phi e_i}|^2\int_0^2|\chi(t)|^2\,\mathbb{E}\left(\left|\int_0^t e^{is\phi(\xi)}d\beta_i(s)\right|^2\right)dt\\ &= \left\||t|^{\frac{1}{2}}\chi\right\|^2_{L_t^2}\sum_{i\in\mathbb{N}}|\hat{\Phi e_i}|^2.\end{aligned} \tag{20}$$

To estimate S_2, we have:

$$
\begin{aligned}
S_2 &= \sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \left[\mathbb{E}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\left| \begin{array}{c} \chi(t_1)\int_0^{t_1} e^{is\phi(\xi)}d\beta_i(s) \\ -\chi(t_2)\int_0^{t_2} e^{is\phi(\xi)}d\beta_i(s) \end{array} \right|^2}{|t_1-t_2|^{1+2b}} dt_1dt_2 \right)\right] \\
&= 2\sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \int_{t_2>0}\int_{t_1<t_2} \frac{\mathbb{E}\left(\left| \begin{array}{c} \chi(t_1)\int_0^{t_1} e^{is\phi(\xi)}d\beta_i(s) \\ -\chi(t_2)\int_0^{t_2} e^{is\phi(\xi)}d\beta_i(s) \end{array} \right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \\
&\leq \sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \Bigg[2\int_{t_2>0}\int_{t_1<0} \frac{|\chi(t_2)|^2\mathbb{E}\left(\left|\int_0^{t_2} e^{is\phi(\xi)}d\beta_i(s)\right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \\
&+2\int_{t_2>0}\int_{0<t_1<t_2} \frac{\mathbb{E}\left(\left| \begin{array}{c} \chi(t_1)\int_0^{t_1} e^{is\phi(\xi)}d\beta_i(s) \\ -\chi(t_2)\int_0^{t_1} e^{is\phi(\xi)}d\beta_i(s) \\ +\chi(t_2)\int_{t_1}^{t_2} e^{is\phi(\xi)}d\beta_i(s) \end{array} \right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \Bigg] \qquad (21)\\
&\leq \sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \Bigg[2\int_{t_2>0}\int_{t_1<0} \frac{|\chi(t_2)|^2\mathbb{E}\left(\left|\int_0^{t_2} e^{is\phi(\xi)}d\beta_i(s)\right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \\
&+4\int_{t_2>0}\int_{0<t_1<t_2} \frac{|\chi(t_1)-\chi(t_2)|^2\mathbb{E}\left(\left|\int_0^{t_1} e^{is\phi(\xi)}d\beta_i(s)\right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \\
&+4\int_{t_2>0}\int_{0<t_1<t_2} \frac{|\chi(t_2)|^2\mathbb{E}\left(\left|\int_{t_1}^{t_2} e^{is\phi(\xi)}d\beta_i(s)\right|^2\right)}{|t_1-t_2|^{1+2b}} dt_1dt_2 \Bigg] \\
&= \sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \left[I_1+I_2+I_3\right].
\end{aligned}
$$

Now, we limit I_1, I_2, and I_3 separately,

$$
I_1 \leq 2\int_0^2 t_1|\chi(t_2)|^2 \int_{t_1<0} \frac{1}{|t_1-t_2|^{1+2b}} dt_1dt_2 \leq M_b \left\| |t|^{\frac{1}{2}-b}\chi \right\|^2_{L_t^2}. \qquad (22)
$$

Using Eq. (15) and the assumption that $2b \in (0,1)$, we have

$$
\begin{aligned}
I_2 &\leq 4\int_0^\infty \int_0^{t_2} \frac{t_1|\chi(t_1)-\chi(t_2)|^2}{\|t_1-t_2\|^{1+2b}} dt_1dt_2 \\
&\leq 4\int_0^2 \int_0^{t_2} \frac{t_1|\chi(t_1)-\chi(t_2)|^2}{\|t_1-t_2\|^{1+2b}} dt_1dt_2 \\
&+ 4\int_2^\infty \int_0^{2} \frac{t_1|\chi(t_1)|^2}{\|t_1-t_2\|^{1+2b}} dt_1dt_2 \\
&\leq 8\|\chi\|^2_{H_t^b} + 4\left\| |t|^{\frac{1}{2}}\chi \right\|^2_{L_t^\infty} \int_0^\infty \int_0^2 \frac{1}{|t_1-t_2|^{1+2b}} dt_1dt_2 \\
&\leq 8\|\chi\|^2_{H_t^b} + M_b\left\| |t|^{\frac{1}{2}}\chi \right\|^2_{L_t^\infty}. \qquad (23)
\end{aligned}
$$

Similarly,

$$
I_3 \leq 4\int_0^2 \int_0^{t_2} \frac{|\chi(t_2)|^2}{|t_1-t_2|^{2b}} dt_1dt_2 \leq M_b \left\| |t|^{\frac{1}{2}-b}\chi \right\|^2_{L_t^2}. \qquad (24)
$$

Combining (20)–(24) with (17), we get

$$
\mathbb{E}\left(\|\mathfrak{F}_x w(.,\xi)\|^2_{H_t^b}\right) \leq N(b,\chi)\sum_{i\in\mathbb{N}} |\hat{\Phi e_i}|^2 \qquad (25)
$$

where $N(b,\chi) = M_b\left(\|\chi\|_{H_t^b} + \||t|^{\frac{1}{2}}\chi\|_{L_t^2} + \||t|^{\frac{1}{2}}\chi\|_{L_t^\infty}\right)$. Hence, the estimate (14) comes from (16) and (25).

Local well-posedness: proof of Theorem 1

According to the stochastic estimation proved in the above section and the Banach fixed point theorem, we deduce a local well-posedness result of Eq. (1). That is, this section is devoted to the proof of Theorem 1. Let $v(t) = U(t)u_0$ and $\bar{u} = u(t) - v(t) - u_l(t)$, then Eq. (5) is equivalent to

$$\bar{u}(t) = \mathcal{A}\bar{u}(t) := \frac{1}{2}\int_0^t U(t-s)\frac{\partial}{\partial x}\left(\bar{u}^2 + u_l^2 + v^2 + 2\left(\bar{u}u_l + \bar{u}v + vu_l\right)\right)(s)ds. \tag{26}$$

Therefore, the goal of this section becomes to prove that $\mathcal{A}$ is a contraction mapping in

$$\mathfrak{Y}_R^T = \left\{\bar{u} \in \mathfrak{X}_{s,b}^T : \|\bar{u}\|_{\mathfrak{X}_{s,b}^T} \le R\right\}, \qquad R > 0, \quad T > 0,$$

where R and T are sufficiently large and small, respectively. Before doing this, we recall some previous results on the linear and bilinear estimates.

Lemma 2 *[23] Assume that $a > 0$, $b < \frac{1}{2}$ and b is close enough to $\frac{1}{2}$. For $s \in \mathbb{R}$, $u_0 \in H^s(\mathbb{R})$, and $f \in \mathfrak{X}_{s,-a}^T$, we have:*

$$\left\|\int_0^t U(t-\tau)f(\tau)d\tau\right\|_{\mathfrak{X}_{s,b}^T} \le CT^{1-a-b}\|f\|_{\mathfrak{X}_{s,b}^T} \tag{27}$$

and

$$\|v\|_{\mathfrak{X}_{s,b}^T} \le \|u_0\|_{H^s}. \tag{28}$$

Lemma 3 *[18] Assume that $a > 0$, $b < \frac{1}{2}$, and b is close enough to $\frac{1}{2}$. For $b' > \frac{1}{2}$, $s > -\frac{7}{4}$, and $u_1, u_2 \in \mathcal{S}(\mathbb{R}^2)$, we have:*

$$\left\|\frac{\partial}{\partial x}(u_1u_2)\right\|_{\mathfrak{X}_{s,-a}} \le C\|u_1\|_{\mathfrak{X}_{s,b}}\|u_1\|_{\mathfrak{X}_{s,b'}} \tag{29}$$

provided that the right hand side is finite.

According to Lemmas 1, 2, and 3, we obtain

$$\|\mathcal{A}\bar{u}\|_{\mathfrak{X}_{s,b}^T} \le C'T^{1-a-b}\left(R^2 + \|u_l\|_{\mathfrak{X}_{s,b}^T} + \|u_0\|_{H^s}\right). \tag{30}$$

Therefore, for $\bar{u}_1, \bar{u}_2 \in \mathfrak{Y}_R^T$, we get

$$\|\mathcal{A}\bar{u}_1 - \mathcal{A}\bar{u}_2\|_{\mathfrak{X}_{s,b}^T} \le C'T^{1-a-b}\left(R^2 + \|u_l\|_{\mathfrak{X}_{s,b}^T} + \|u_0\|_{H^s}\right)\|\bar{u}_1 - \bar{u}_2\|_{\mathfrak{X}_{s,b}^T}. \tag{31}$$

Now, define the stopping time T_ω by:

$$T_\omega = \inf\left\{t > 0 : 4C't^{1-a-b}R_\omega^T \ge 1\right\}, \tag{32}$$

where $R_\omega^T = \|u_l\|_{\mathfrak{X}_{s,b}^T} + \|u_0\|_{H^s}$. Then, $\mathcal{A}$ maps the ball with center zero and radius R_ω^T in $\mathfrak{X}_{s,b}^{T_\omega}$ into itself, and

$$\|\mathcal{A}\bar{u}_1 - \mathcal{A}\bar{u}_2\|_{\mathfrak{X}_{s,b}^{T_\omega}} \le \frac{3}{4}\|\bar{u}_1 - \bar{u}_2\|_{\mathfrak{X}_{s,b}^{T_\omega}}. \tag{33}$$

From the fixed point theory, $\mathcal{A}$ has a unique fixed point, which is the solution of (5) in $\mathfrak{X}_{s,b}^{T_\omega}$. Observe that $u = v + \bar{u} + u_l \in \mathfrak{X}_{s,b'}^{T_\omega} + \mathfrak{X}_{s,b}^{T_\omega}$.

In the remaining part of this section, we complete the proof by showing that $u \in C([0,T_\omega], H^s(\mathbb{R}))$. Taking in attention that $b < \frac{1}{2}, b' > \frac{1}{2}$. By virtue of the Sobolev imbedding theorem, we have $v \in C([0,T_\omega], H^s(\mathbb{R}))$. Under the condition that $\Phi \in L_2^{0,s}$ and the fact that $U(t)$ is a unitary group in $H^s(\mathbb{R})$, an application of Theorem 6.10 in [16] implies that $u_l \in C([0,T_\omega]; H^s(\mathbb{R}))$.

Now, choose a cutoff function $\chi_T \in C_0^\infty(\mathbb{R})$ such that $\chi_T(t) = 1$ on $[0,2]$, supp $\chi_T \subset [-1,2]$, and $\chi_T(t) = 0$ on $(-\infty,-1] \cup [2,\infty)$. Denote $\chi_q(.) = \chi\left(q^{-1}(.)\right)$ for some $q \in \mathbb{R}$. By Lemma 3, we have $\tilde{u}\tilde{u}_x \in \mathfrak{X}_{s,-a}$ for any prolongation $\tilde{u}$ of u in $\mathfrak{X}_{s,c} + \mathfrak{X}_{s,b}$. Therefore,

$$\left\| \chi_T \int_0^t U(t-s)\left(\tilde{u}(s)\tilde{u}_x(s)\right) \right\|_{\mathfrak{X}_{s,1-a}} \le C \left\| \tilde{u}(s)\tilde{u}_x(s) \right\|_{\mathfrak{X}_{s,-a}}. \tag{34}$$

Since $1-a > \frac{1}{2}$, then $\tilde{u} \in \mathfrak{X}_{s,b} \subset C([0,T_\omega]; H^s(\mathbb{R}))$. This completes the proof of Theorem 1.

Global well-posedness: proof of Theorem 2

Fix $T_0 > 0$ and assume that u_0 satisfies the conditions of Theorem 1. In this section, we present a proof of Theorem 2, that is, we show that the solution u can be extended to the whole interval $[0,T_0]$. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $L_0^{0,4}$ such that

$$\lim_{n\to\infty} \Phi_n = \Phi \quad \text{in } L_2^{0,0}. \tag{35}$$

and let $\left(u_{0,n}\right)_{n\in\mathbb{N}}$ be another sequence in $L^2(\Omega, H^s(\mathbb{R}))$ such that

$$\lim_{n\to\infty} u_{0,n} = u_0 \quad \text{in } L^2\left(\Omega, L^2(\mathbb{R})\right). \tag{36}$$

By using reasoning similar to that in [23], we can find a unique solution u_n in $C\left([0,T_0], H^3(\mathbb{R})\right)$ for

$$u_n = U(t)u_{0,n} + \int_0^t U(t-s)\left(u_n(s)\frac{\partial u_n}{\partial x}(s)\right)ds + \int_0^t U(t-s)\Phi_n dW(s). \tag{37}$$

By using the Itô formula on $\|u_n\|^2_{L^2(\mathbb{R})}$ and martingale inequality (see [16]), we have

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} \|u_n\|^2_{L_x^2}\right) \le \mathbb{E}\left(\|u_{0,n}\|^2_{L_x^2}\right) + C\|\Phi_n\|^2_{L_2^{0,0}}. \tag{38}$$

Therefore, the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded and weakly star convergent to a function $u^* \in L^2\left(\Omega; L^\infty\left([0,T_0]; L^2(\mathbb{R})\right)\right)$, which satisfies

$$\mathbb{E}\left(\sup_{t\in[0,T_0]} \|u^*\|^2_{L_x^2}\right) \le \mathbb{E}\left(\|u_0\|^2_{L_x^2}\right) + C\|\Phi\|^2_{L_2^{0,0}}. \tag{39}$$

In the same way as $\mathcal{A}$, define the mapping $\mathcal{A}_n$. It is easy to show that $\mathcal{A}_n$ is uniformly strict contraction on $\mathfrak{Y}_{r(\omega)}^{t(\omega)}$ in $\mathfrak{X}_{s,b}^{T_\omega}$. According to the fixed point theorem, there exists a unique function $u \in \mathfrak{X}_{s,b}^{T_\omega}$ such that

$$u = u^* = \lim_{n\to\infty} u_n \quad \text{a.s. in } [0,T_\omega], \tag{40}$$

where u_n is the unique fixed point of $\mathcal{A}_n$. Also, we have

$$\|u(t(\omega))\|_{L^2(\mathbb{R})} \le \|u^*\|_{L^\infty([0,T_0];L^2(\mathbb{R}))}. \tag{41}$$

Thus, we can emerge a solution on $[T_\omega, 2T_\omega]$. Hence, the solution u can be extended to $[0,T_0]$ almost surely by reiteration. This completes the proof of Theorem 2.

Summary and discussion

This paper is devoted to employ the Fourier restriction method, the Banach contraction principle, and some basic inequalities for investigating nonlinear SPDEs and for proving local and global well-posedness results for their solutions in convenient function spaces. Our attention is focused on the stochastic Kawahara Eq. (1), which is a fifth-order shallow water wave equation considered in random environment. We prove that Eq. (1) is locally well-posed for data in $H^s(\mathbb{R})$, $s > -7/4$ and its solution can be extended to a global one on $[0, T_0]$. The Fourier restriction method is proposed due to the non-zero singularity of the phase function ϕ.

The deterministic Kawahara Eq. (7) was discussed by Jia and Huo in [18]. They proved local well-posedness result for data in $H^s(\mathbb{R})$, $s > -7/4$. In this paper, we extend their result and handle the stochastic version of the Kawahara equation by choosing new appropriate stochastic function spaces (such as the space $\mathfrak{X}^T_{s,b}$) and estimating the stochastic convolution (12) in these spaces. That is, we consider a realistic situation of the fifth-order shallow water wave equations. We believe that the ideas which we have suggested in this paper can be also applied to a wide class of stochastic nonlinear evolution equations in the field of mathematical physics, for instance, the stochastic modified Kawahara, generalized KdV, Hirota-Satsuma coupled KdV, and Swada-Kotera equations.

Abbreviations
KdV: Korteweg-de Vries; SPDEs: Stochastic partial differential equations

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