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Balanced factor congruences of double *MS*-algebras

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Abstract
In this paper, we introduce the concepts of Stone elements, central elements and Birkhoof central elements of a double *MS*-algebra and study their related properties. We observe that the center $C(L)$ of a double MS -algebra L is precisely the Birkhoof center $BC(L)$ of L. A complete description of factor congruences on a double *MS*-algebra is given by means of the central elements. A characterization of balanced factor congruences of double *MS*-algebra is obtained. A one-to-one correspondence between the class of all balanced factor congruences of a double *MS*-algebra L and the central elements of L is obtained.

Keywords: De Morgan algebra, *MS*-algebra, Double *MS*-algebra, Factor congruences, Balanced factor congruences

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Introduction

Blyth and Varlet [1] introduced *MS*-algebras as a generalization of both de Morgan algebras and Stone algebras. Blyth and Varlet [2] characterized the subvarieties of the class **MS** of all *MS*-algebras. Badawy, Guffova, and Haviar [3] introduced and characterized the class of principal *MS*-algebras and the class of decomposable *MS*-algebras by means of triples. Badawy [4] introduced and studied many properties of d_L-filters of principal *MS*-algebras. Also, Badawy and El-Fawal [5] considered homomorphisms and subalgebras of decomposable *MS*-algebras.

Blyth and Varlet [6] introduced the class of double *MS*-algebras and showed that every de Morgan algebra M can be represented non-trivially as the skeleton of the double *MS*-algebra $M^{[2]} = \{(a,b) \in M \times M : a \leq b\}$. The class of double *MS*-algebras satisfying the complement property has been introduced by Congwen [7]. Haviar [8] studied affine complete of double *MS*-algebras from the class $\mathbf{K_2}$, of all double K-algebras. Wang [9] introduced the notion of congruence pairs of double K_2-algebras. Recently, Badawy [10] introduced and constructed the class of double *MS*-algebras satisfying the generalized complement property that is containing the class of double MS -algebras satisfying the complement property.

In this paper, we introduce the notion of Stone elements in double MS -algebras. Then, we prove that the set of Stone elements of a double *MS*-algebra L forms the greatest Stone subalgebra of L. We introduce the concept of central elements of a double *MS*-algebra L

and we show that the set $C(L)$ of all central elements forms the greatest Boolean subalgebra of L. For a principal ideal $(a]$ (filter $[a)$) of a double MS-algebra $(L;^{\circ},^{+})$, it is observed that a relativized algebra $(a]_L = ((a];\vee,\wedge,^{\circ_a},^{+_a},0,a)$ $([a)_L = ([a);\vee,\wedge,^{\circ_a},^{+_a},a,1))$ is a double MS-algebra if and only if a is a central element of L, where $x^{\circ_a} = x^{\circ} \wedge a$ and $x^{+_a} = x^{+} \wedge a$ ($x^{\circ_a} = x^{\circ} \vee a$ and $x^{+_a} = x^{+} \vee a$). Also, we introduce the Birkhoof center of a double MS-algebra, then we showed that the Birokhoof center of a double MS-algebra L can be identified with the center of L. Factor congruences of a double MS-algebra are investigated by means of central elements. Finally, we study and characterize balanced factor congruences of a double MS-algebra. There is one-to-one correspondence between the class of balanced factor congruences of a double MS-algebra L and the center $C(L)$ of L.

Preliminaries

In this section, some definitions and results were introduced in [1, 2, 6, 11, 12].

A de Morgan algebra is an algebra $(L;\vee,\wedge,^{-},0,1)$ of type (2,2,1,0,0) where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and $^{-}$ the unary operation of involution satisfies:

$\overline{\overline{x}} = x, \overline{(x\vee y)} = \overline{x}\wedge\overline{y}, \overline{(x\wedge y)} = \overline{x}\vee\overline{y}.$

An MS-algebra is an algebra $(L;\vee,\wedge,^{\circ},0,1)$ of type (2,2,1,0,0) where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and a unary operation $^{\circ}$ satisfies:

$x \leq x^{\circ\circ}, (x\wedge y)^{\circ} = x^{\circ}\vee y^{\circ}, 1^{\circ} = 0.$

The basic properties of MS-algebras are given in the following theorem.

Theorem 1 (Blyth and Varlet [6]) *For any two elements a, b of an MS-algebra L, we have*
(1) $0^{\circ\circ} = 0$ *and* $1^{\circ\circ} = 1$,
(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$,
(3) $a^{\circ\circ\circ} = a^{\circ}$,
(4) $a^{\circ\circ\circ\circ} = a^{\circ\circ}$,
(5) $(a\vee b)^{\circ} = a^{\circ}\wedge b^{\circ}$,
(6) $(a\vee b)^{\circ\circ} = a^{\circ\circ}\vee b^{\circ\circ}$,
(7) $(a\wedge b)^{\circ\circ} = a^{\circ\circ}\wedge b^{\circ\circ}$.

A dual MS-algebra is an algebra $(L;\vee,\wedge,^{+},0,1)$ of type (2,2,1,0,0) where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and the unary operation $^{+}$ satisfies:

$x \geq x^{++}, (x\wedge y)^{+} = x^{+}\vee y^{+}, 0^{+} = 1.$

Proposition 1 *For any two elements a, b of a dual MS-algebra $(L;^{+})$, we have*
(1) $0^{++} = 0$ *and* $1^{++} = 1$,
(2) $a \leq b \Rightarrow b^{+} \leq a^{+}$,
(3) $a^{+++} = a^{+}$,
(4) $a^{++++} = a^{++}$,
(5) $(a\vee b)^{+} = a^{+}\wedge b^{+}$,
(6) $(a\vee b)^{++} = a^{++}\vee b^{++}$,
(7) $(a\wedge b)^{++} = a^{++}\wedge b^{++}$.

A double MS-algebra is an algebra $(L;^{\circ},^{+})$ such that $(L;^{\circ})$ is an MS-algebra, $(L;^{+})$ is a dual MS-algebra, and the unary operations $^{\circ},^{+}$ are linked by the identities $x^{+\circ} = x^{++}$ and $x^{\circ+} = x^{\circ\circ}$, for all $x \in L$.

For any element x of a double MS-algebra L, it is clear that $x^{++} \leq x^{\circ\circ}$ and consequently $x^{\circ} \leq x \leq x^{+}$.

Some subsets of a double MS-algebra play a significant role in the investigation, by the skeleton $L^{\circ\circ}$ of a double MS-algebra L we mean a de Morgan algebra

$L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\} = L^{++} = \{x \in L : x = x^{++}\} = \{x \in L : x^{\circ} = x^{+}\}$.

An equivalence relation θ on a lattice L is called a lattice congruence on L if $(a, b) \in \theta$ and $(c, d) \in \theta$ implies $(a \vee c, b \vee d) \in \theta$ and $(a \wedge c, b \wedge d) \in \theta$.

Theorem 2 (Blyth [12]) *An equivalence relation on a lattice L is a lattice congruence on L if and only if* $(a, b) \in \theta$ *implies* $(a \vee z, b \vee z) \in \theta$ *and* $(a \wedge z, b \wedge z) \in \theta$ *for all* $z \in L$.

A lattice congruence θ on a double MS-algebra $(L;^{\circ}, ^{+})$ is called a congruence on L if $(a, b) \in \theta$ implies $(a^{\circ}, b^{\circ}) \in \theta$ and $(a^{+}, b^{+}) \in \theta$.

We use $\nabla = L \times L$ for the universal congruence on a lattice L and $\Delta = \{(a, a) : a \in L\}$ for the equality congruence on L.

We say the congruences θ, ψ on a lattice L are permutable if $\theta \circ \psi = \psi \circ \theta$, that is, $x \equiv y(\theta)$ and $y \equiv z(\psi)$ imply $x \equiv r(\psi)$ and $r \equiv z(\theta)$ for some $y, r \in L$.

Center and Birkhoof center of a double *MS*-algebra

We introduce the concept of Stone elements of a double MS-algebra L. Then, we show that the set L_S of all Stone elements of L is the greatest Stone subalgebra of L.

Definition 1 *An element x of a double MS-algebra L is called a Stone element of L if* $x^{\circ} \vee x^{\circ\circ} = 1$ *and* $x^{+} \wedge x^{++} = 0$. *Let* L_S *denote the set of all Stone elements of L, that is,* $L_S = \{x \in L : x^{\circ} \vee x^{\circ\circ} = 1, x^{+} \wedge x^{++} = 0\}$.

Definition 2 *Let* L_1 *be a bounded sublattice of a double MS-algebra L . Then,* L_1 *is called a subalgebra of L if* $x^{\circ}, x^{+} \in L_1$ *for every* $x \in L_1$.

Definition 3 *A subalgebra* L_1 *of a double MS-algebra L is called a Stone subalgebra if* $x^{\circ} \vee x^{\circ\circ} = 1$ *and* $x^{+} \wedge x^{++} = 0$, *for all* $x \in L_1$.

Proposition 2 L_S *is the greatest Stone subalgebra of a double MS-algebra L.*

Proof It is clear that $0, 1 \in L_S$. Let $x, y \in L_S$. Then, $x^{\circ} \vee x^{\circ\circ} = 1$, $x^{+} \wedge x^{++} = 0$, $y^{\circ} \vee y^{\circ\circ} = 1$, and $y^{+} \wedge y^{++} = 0$. Thus, we get

$$
\begin{aligned}
(x \vee y)^{\circ} \vee (x \vee y)^{\circ\circ} &= (x^{\circ} \wedge y^{\circ}) \vee (x^{\circ\circ} \vee y^{\circ\circ}) \text{ by Theorem 1(5),(6)} \\
&= (x^{\circ} \vee x^{\circ\circ} \vee y^{\circ\circ}) \wedge (y^{\circ} \vee x^{\circ\circ} \vee y^{\circ\circ}) \text{ by distributivity of } L \\
&= 1 \wedge 1 = 1 \text{ as } x^{\circ} \vee x^{\circ\circ} = 1, y^{\circ} \vee y^{\circ\circ} = 1, \\
(x \vee y)^{+} \wedge (x \vee y)^{++} &= (x^{+} \wedge y^{+}) \wedge (x^{++} \vee y^{++}) \text{ by Proposition 1(5),(6)} \\
&= (x^{+} \wedge y^{+} \wedge x^{++}) \vee (x^{+} \wedge y^{+} \wedge y^{++}) \text{ by distributivity of } L \\
&= 0 \vee 0 = 0 \text{ as } x^{+} \wedge x^{++} = 0, y^{+} \wedge y^{++} = 0.
\end{aligned}
$$

Then, $x \vee y \in L_S$. Using a similar way, we get $x \wedge y \in L_S$. Therefore, $(L_S, \vee, \wedge, 0, 1)$ is a bounded distributive sublattice of L. Now, we prove that $x^+ \in L_S$ for all $x \in L_S$.

$$\begin{aligned} x^{+\circ} \vee x^{+\circ\circ} &= x^{++} \vee x^{+++} \text{as } x^{+\circ} = x^{++} \\ &= (x^{+} \wedge x^{++})^{+} \text{ by Proposition 1(5)} \\ &= 0^{+} = 1 \text{ as } x^{+} \wedge x^{++} = 0, \\ x^{+\circ} \wedge x^{+\circ\circ} &= x^{++} \wedge x^{+++} \text{ as } x^{+\circ} = x^{++} \\ &= x^{++} \wedge x^{+} = 0 \text{ by Proposition 1(3).} \end{aligned}$$

Hence, $x^+ \in L_S$. Similarly, we can prove that $x^\circ \in L_S$ for all $x \in L_S$. Therefore, L_S is a subalgebra of a double MS-algebra L. Since $x^\circ \vee x^{\circ\circ} = 1$ and $x^+ \wedge x^{++} = 0$ for every $x \in L_S$, then L_S is a Stone subalgebra of L. To prove that L_S is the greatest Stone subalgebra of L, let S be any Stone subalgebra of L. Let $x \in S$. Then, x is a Stone element of L, and hence, $x \in L_S$. So $S \subseteq L_S$ as claimed. □

On the following, we introduce the notion of central elements of a double MS-algebra L and prove that the set $C(L)$ of all central elements of L is the greatest Boolean subalgebra of L. Also, the relationship among L_S, $C(L)$, and $L^{\circ\circ}$ is investigated.

Definition 4 *An element a of double MS-algebra L is called a central element if* $x \vee x^\circ = 1$ *and* $x \wedge x^+ = 0$*. The set of all central elements of L is called the center of L and is denoted by* $C(L)$*, that is,* $C(L) = \{x \in L : x \vee x^\circ = 1, x \wedge x^+ = 0\}$.

Example 1 *Consider the bounded distributive lattice L in Fig. 1. Define unary operations* $^\circ, ^+$ *on L by*

$$b^\circ = x^\circ = a, d^\circ = y^\circ = c, 1^\circ = z^\circ = 0, 0^\circ = 1, c^\circ = d, a^\circ = b \tag{1}$$

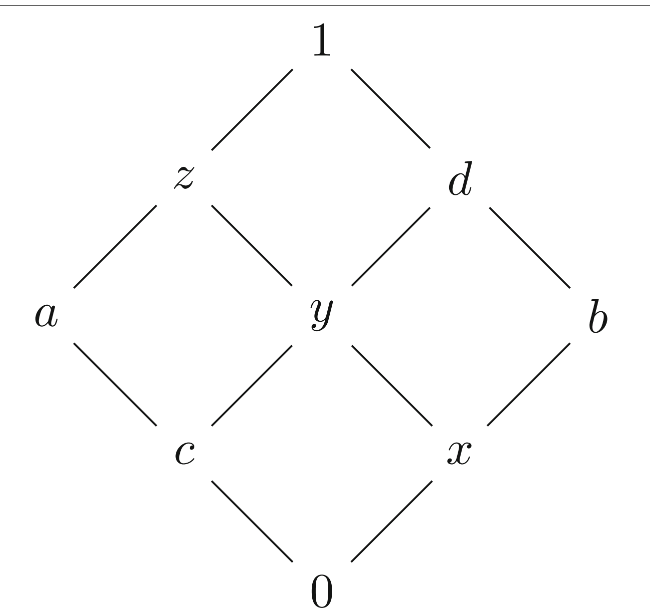

Fig. 1 *L*

and

$$a^+ = z^+ = b, c^+ = y^+ = d, 0^+ = x^+ = 1, b^+ = a, d^+ = c, 1^+ = 0. \tag{2}$$

It is clear that $(L;^\circ,^+)$ *is a double MS-algebra. Then,* $L^{\circ\circ}, L_S$*, and* $C(L)$ *are given in Figs. 2, 3, and 4, respectively.*

Theorem 3 *Let L be a double MS-algebra. Then*
(1) $C(L) = L^{\circ\circ} \cap L_S$*,*
(2) $C(L)$ *is the greatest Boolean subalgebra of L,* L_S*, and* $L^{\circ\circ}$*,*
(3) $C(L) = C(L_S) = C(L^{\circ\circ})$*.*

Proof (1). Let $x \in C(L)$. Then, $x \vee x^\circ = 1$ and $x \wedge x^+ = 0$. Then

$$\begin{aligned} x^{++} &= x^{++} \vee 0 \\ &= x^{++} \vee (x \wedge x^+) \\ &= (x^{++} \vee x) \wedge (x^{++} \vee x^+) \text{ by distributivity of } L \\ &= x \wedge (x^+ \wedge x)^+ \text{ as } x \geq x^{++} \\ &= x \wedge 0^+ = x \wedge 1 = x. \end{aligned}$$

Thus, $x \in L^{\circ\circ}$. Also,

$$\begin{aligned} x^{++} \wedge x^+ &= x^{++} \wedge x^{+++} \text{ by Proposition 1(3)} \\ &= (x \wedge x^+)^{++} \text{ by Proposition 1(7)} \\ &= 0^{++} = 0 \text{ by Proposition 1(1),} \\ x^{\circ\circ} \vee x^\circ &\geq x \vee x^\circ \text{ as } x^{\circ\circ} \geq x \\ &= 1. \end{aligned}$$

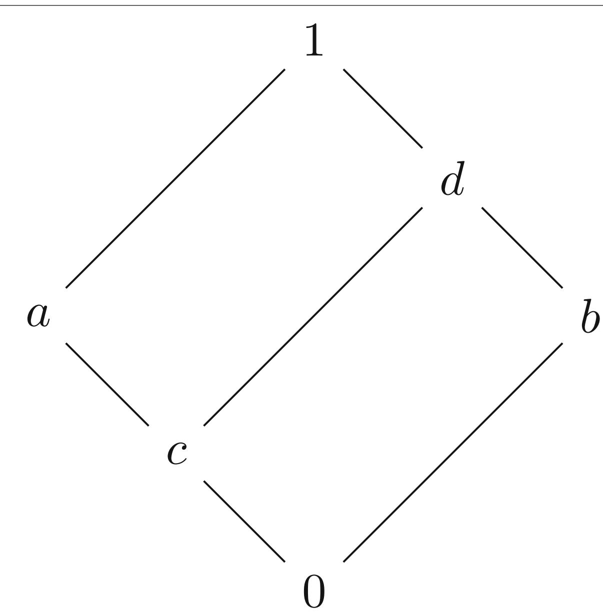

Fig. 2 $L^{\circ\circ}$

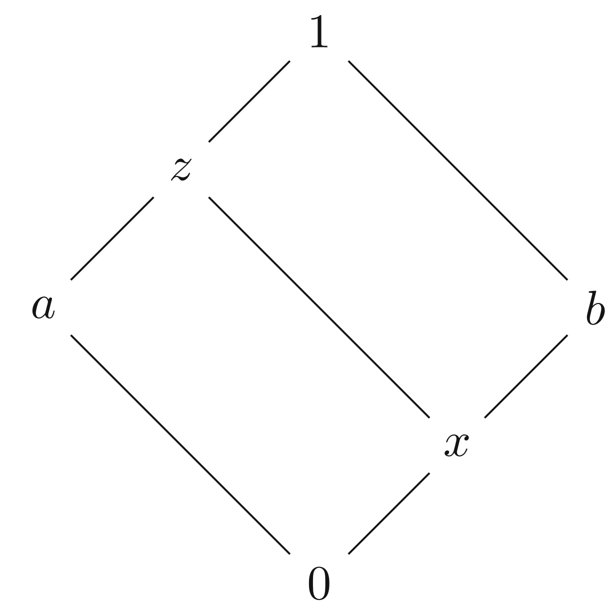

Fig. 3 L_S

Then, $x^{++} \wedge x^{+} = 0$ and $x^{\circ\circ} \vee x^{\circ} = 1$ imply $x \in L_S$. Therefore, $C(L) \subseteq L^{\circ\circ} \cap L_S$. Conversely, let $x \in L^{\circ\circ} \cap L_S$. Then, $x = x^{\circ\circ} = x^{++}$, $x^{\circ} \vee x^{\circ\circ} = 1$, and $x^{+} \wedge x^{++} = 0$. Now,

$$\begin{aligned} x^{\circ} \vee x &= x^{\circ} \vee x^{\circ\circ} = 1, \\ x \wedge x^{+} &= x^{++} \wedge x^{+} = 0. \end{aligned}$$

Thus, $x \in C(L)$, and hence, $L^{\circ\circ} \cap L_S \subseteq C(L)$.

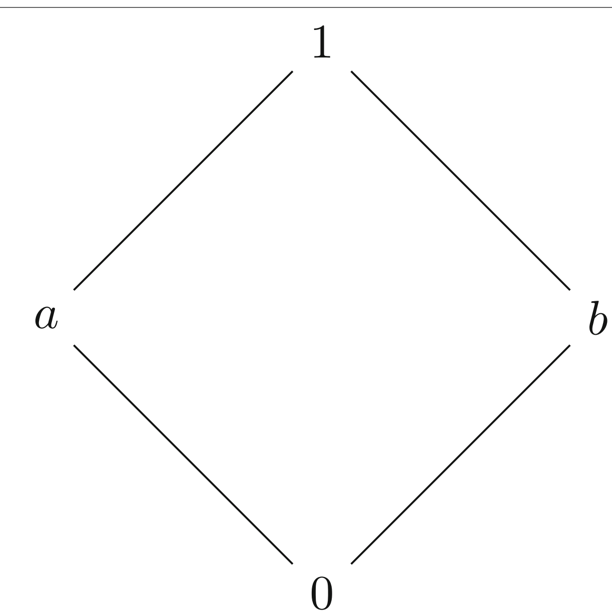

Fig. 4 $C(L)$

(2) Clearly $0, 1 \in C(L)$. Let $a, b \in C(L)$. Then, we have

$$\begin{aligned}
(a \vee b) \vee (a \vee b)^{\circ} &= a \vee b \vee (a^{\circ} \wedge b^{\circ}) \text{ by Proposition 1(5)}\\
&= (a \vee b \vee a^{\circ}) \wedge (a \vee b \vee b^{\circ}) \text{ by distributivity of } L\\
&= 1 \wedge 1 = 1,\\
(a \vee b) \wedge (a \vee b)^{+} &= (a \vee b) \wedge (a^{+} \wedge b^{+}) \text{ by Theorem 1(5)}\\
&= (a \wedge a^{+} \wedge a^{+}) \vee (b \wedge a^{+} \wedge b^{+}) \text{ by distributivity of } L\\
&= 0 \vee 0 = 0.
\end{aligned}$$

Then, $a \vee b \in C(L)$. Similarly $a \wedge b \in C(L)$. Therefore, $(C(L); \vee, \wedge, 0, 1)$ is a bounded sublattice of L. Now, we observe that $a^{\circ} \in C(L)$ for all $a \in C(L)$,

$$\begin{aligned}
a^{\circ\circ} \vee a^{\circ\circ\circ} &= a \vee a^{\circ} \text{ as } a^{\circ\circ} = a, \forall a \in C(L) \text{ and } a^{\circ\circ\circ} = a^{\circ}\\
&= 1,\\
a^{\circ+} \wedge a^{\circ++} &= a^{\circ\circ} \wedge a^{\circ\circ\circ} \text{ as } a^{\circ+} = a^{\circ\circ}\\
&= a^{\circ\circ} \wedge a^{\circ} \text{ as } a^{\circ\circ\circ} = a^{\circ}\\
&= (a^{\circ} \vee a)^{\circ} = 1^{\circ} = 0 \text{ by Theorem 1(5)}.
\end{aligned}$$

Since $a^{\circ} = a^{+}$ for all $a \in C(L)$, then $^{\circ}$ coincide with $^{+}$ on $C(L)$. Therefore, $(C(L), \vee, \wedge,^{\circ}, 0, 1)$ is a subalgebra of L. Since $a \vee a^{\circ} = 1$ and $a \wedge a^{\circ} = a^{\circ\circ} \wedge a^{\circ} = (a^{\circ} \vee a)^{\circ} = 1^{\circ} = 0$ for all $a \in C(L)$, then $(C(L), \vee, \wedge,^{\circ}, 0, 1)$ is a Boolean subalgebra of L. Suppose that B is any Boolean subalgebra of L and $x \in B$. Then, $a \vee a^{\circ} = 1$ and $a \wedge a^{+} = a \wedge a^{\circ} = 0$. Hence, a is a central element of L and $a \in C(L)$. So, $B \subseteq C(L)$ and $C(L)$ is the greatest Boolean subalgebra of L. Using similar agrement, we can show that $C(L)$ is also the greatest Boolean subalgebra of both L_S and $L^{\circ\circ}$.
(3) It follows (1) and (2). □

The following theorem shows that the centers of isomorphic double *MS*-algebras are isomorphic Boolean algebras.

Theorem 4 *If L and M are isomorphic double MS-algebras, then their centers are isomorphic.*

Proof Let $h : L \to M$ be an isomorphism and $a \in C(L)$. Then, $a \vee a^{\circ} = 1$ and $a \wedge a^{+} = 0$. Hence, $h(a \vee a^{\circ}) = h(a) \vee h(a^{\circ}) = h(a) \vee (h(a))^{\circ} = h(1) = 1$ and $h(a) \wedge (h(a))^{+} = h(0) = 0$. Therefore, $h_{C(L)}(a) = h(a) \in C(M)$. It is clear that $h_{C(L)}$ is an injective (0,1) lattice homomorphism. Let $b \in C(M)$. Then, there exists $b \in L$ such that $b = h(a) = h_{C(L)}(a)$ as h is onto. It follows that $b^{\circ\circ} = (h(a))^{\circ\circ} = h(a^{\circ\circ}) = h(a) = h_{C(L)}(a)$. Thus, $h_{C(L)}$ is onto. Obviously, $h_{C(L)}$ preserves $^{\circ}$ and $^{+}$. Then, $h_{C(L)}$ is an isomorphism, and hence, $C(L) \cong C(M)$. □

For an *MS*-algebra $(L,^{\circ})$, it is proved in [3] that $(a]_L = ((a],^{\circ_a})$ is an *MS*-algebra if and only if $a^{\circ} \in C(L)$, where $(a] = \{x \in L : x \leq a\} = [0, a]$ is a principal ideal of L generated by the element a of L, a unary operation $^{\circ_a}$ is defined on $(a]$ by $x^{\circ_a} = x^{\circ} \wedge a$ for all $x \in (a]$ and $C(L) = \{x \in L : x \vee x^{\circ} = 1\}$ is the center of L.

For a double *MS*-algebra $(L;^{\circ},^{+})$, the answer of the following question is given: Under what conditions a principal ideal $(a], a \in L$ constructs a double *MS*-algebra?

Theorem 5 *Let L be a double MS-algebra. Suppose that $a \in C(L)$, then the relativized algebra $(a]_L = ((a], \wedge, \vee, ^{\circ_a}, ^{+_a}, a, 1)$ is a double MS-algebra, where $x^{\circ_a} = x^\circ \wedge a$ and $x^{+_a} = x^+ \wedge a$. Conversely, if $(a]_L = ((a], \wedge, \vee, ^{\circ_a}, ^{+_a}, a, 1)$ is a double MS-algebra, then $a \in L_S$.*

Proof Assume that $a \in C(L)$. Hence, $a^\circ \in C(L)$. Then, by ([13], Theorem 3.5), $((a], \vee, \wedge, ^{\circ_a}, 0, a)$ is an MS-algebra, whenever $x^{\circ_a} = x^\circ \wedge a$. Now, we prove that $((a], \vee, \wedge, ^{+_a}, 0, a)$ is a dual MS-algebra, where $x^{+_a} = x^+ \wedge a$ for any $x \in (a]$. Let $x \in (a]$, we have

$$\begin{aligned}
x^{+_a \circ +_a} \vee x &= (x^+ \wedge a)^{+_a} \vee x \\
&= ((x^+ \wedge a)^+ \wedge a) \vee x \\
&= ((x^{++} \vee a^+) \wedge a) \vee x \\
&= (x^{++} \wedge a) \vee (a^+ \wedge a) \vee x \text{ by distributivity of } L \\
&= (x^{++} \wedge a) \vee x \text{ as } a^+ \wedge a = 0 \\
&= x \text{ as } x \geq x^{++} \geq x^{++} \wedge a.
\end{aligned}$$

Then, $x \geq x^{+_a +_a}$. Let $x, y \in (a]$

$$\begin{aligned}
(x \wedge y)^{+_a} &= (x \wedge y)^\circ \wedge a \\
&= (x^+ \vee y^{\circ +}) \wedge a \\
&= (x^+ \wedge a) \vee (y^+ \wedge a) \text{ by distributivity of } L \\
&= x^{+_a} \vee y^{+_a},
\end{aligned}$$

Also, $0^{+_a} = a$. Now, for every $x \in (a]$, we have

$$\begin{aligned}
x^{\circ_a +_a} &= (x^\circ \wedge a)^{+_a} \\
&= (x^\circ \wedge a)^+ \wedge a \\
&= (x^{\circ +} \vee a^+) \wedge a \\
&= (x^{\circ\circ} \vee a^+) \wedge a \\
&= (x^{\circ\circ} \wedge a) \vee (a^+ \wedge a) \\
&= x^{\circ\circ} \wedge a \text{ as } a^+ \wedge a = 0, \\
x^{\circ_a \circ_a} &= (x^\circ \wedge a)^\circ \wedge a \\
&= (x^{\circ\circ} \vee a^\circ) \wedge a \\
&= (x^{\circ\circ} \wedge a) \vee (a^\circ \wedge a) \text{ by distributivity} \\
&= x^{\circ\circ} \wedge a \text{ as } a^+ \wedge a = 0.
\end{aligned}$$

This deduce that $x^{\circ_a +_a} = x^{\circ\circ}$. Also, we can get $x^{+_a \circ_a} = x^{+_a +_a}$. Therefore, $(a]_L = ((a], \vee, \wedge, ^{\circ_a}, ^{+_a}, 0, a)$ is a double MS-algebra.
Conversely, suppose that $a \in L$, $(a]_L = ((a], \vee, \wedge, ^{\circ_a}, ^{+_a}, 0, a)$ is a double MS-algebra with $x^{\circ_a} = x^\circ \wedge a$ and $x^{+_a} = x^+ \wedge a$. Since a is the greatest element of $(a]_L$, then $a^{+_a} = 0$ and $a^{\circ_a} = 0$. This gives $a^+ \wedge a = 0$ and $a^\circ \wedge a = 0$, respectively. Consequently, $a^+ \wedge x^{++} = (a^+ \wedge a)^{++} = 0^{++} = 0$ and $a^{\circ\circ} \vee a^\circ = (a^\circ \wedge a)^\circ = 0^\circ = 1$. Therefore, a is a Stone element of L. □

Similarly for the principal filter $[a)$ of a double MS-algebra, we establish the following result, where $[a) = \{x \in L : x \geq a\} = [a, 1]$.

Theorem 6 *Let L be a double MS-algebra. If* $a \in C(L)$*, then the relativized algebra* $[a)_L = ([a), \wedge, \vee, ^{\circ_a}, ^{+_a}, a, 1)$ *is a double MS-algebra, where* $x^{\circ_a} = x^{\circ} \vee a$ *and* $x^{+_a} = x^{+} \vee a$ *. Conversely, if* $[a)_L = ([a), \wedge, \vee, ^{\circ_a}, ^{+_a}, a, 1)$ *is a double MS-algebra, then* $a \in L_S$.

Let L_1, L_2 are double MS-algebras. Then, $L_1 \times L_2$ is a double MS-algebra, where $^{\circ}$ and $^{+}$ are defined by $(x, y)^{\circ} = (x^{\circ}, y^{\circ})$ and $(x, y)^{+} = (x^{+}, y^{+})$. Moreover, $(L_1 \times L_1)^{\circ\circ} = L_1^{\circ\circ} \times L_2^{\circ\circ}$ and $C(L_1 \times L_2) = C(L_1) \times C(L_2)$.

As a consequence of Theorem 5 and Theorem 6, we have

Theorem 7 *Let L be a double MS-algebra. If* $a \in C(L)$*, then* $((a]_L \times [a)_L, ^{\circ}, ^{+})$ *is a double MS-algebra, where*

$(a]_L \times [a)_L = \{(x, y) : x \in (a]_L, y \in [a)_L\}$,

and

$(x, y)^{\circ} = (x^{\circ} \wedge a, y^{\circ} \vee a)$ *and* $(x, y)^{+} = (x^{+} \wedge a, y^{+} \vee a)$ *for all* $(x, y) \in (a]_L \times [a)_L$.

Now, we introduce the concept of Birkhoff center for a double MS-algebra.

Definition 5 *An element a of a double MS-algebra L is called a Birkhoff central element if there exist double MS-algebras* L_1 *and* L_2 *and an isomorphism from L to* $L_1 \times L_2$ *such that a is mapped to* $(1, 0)$*. The set* $BC(L)$ *of all Birkhoff central elements of L is called the Birkhoff center.*

Theorem 8 *Let L be a double MS-algebra. Then,* $BC(L) = C(L)$.

Proof Let $a \in BC(L)$. Then, there exist double MS-algebras L_1 and L_2 and an isomorphism h from L to $L_1 \times L_2$ such that $h(a) = (1, 0)$. By Theorem 4, $C(L)$ is isomorphic to $C(L_1 \times L_1) = C(L_1) \times C(L_2)$. Thus, $(1, 0) \in C(L_1) \times C(L_2)$. Therefore, $a = h^{-1}(1, 0) \in C(L)$ and $BC(L) \subseteq C(L)$.

Conversely, let $a \in C(L)$. Then, by Theorem 5 and Theorem 6, $L_1 = (a]_L$ and $L_2 = [a)_L$ are double MS-algebras, respectively. The direct product $L_1 \times L_2 = (a]_L \times [a)_L$ is a double MS-algebra, by Theorem 7. Notice that $1_{L_1} = a$ is the greatest element of L_1 and $0_{L_2} = a$ is the smallest element of L_2. Define $h : L \to L_1 \times L_2$ by $h(x) = (a \wedge x, a \vee x)$. It is already seen that h is an isomorphism of L onto $L_1 \times L_2$. Then, $h(a) = (a, a) = (1_{L_1}, 0_{L_2})$ implies $a \in BC(L)$. Therefore, $C(L) \subseteq BC(L)$. □

Balanced factor congruences of a double *MS*-algebra

In [14], Badawy investigated the relationship between congruences and de Morgan filters of decomposable MS-algebras. In this section, we study the connection between congruences and central elements of a double MS-algebra.

Let a be an element of a double MS-algebra L. Define a binary relation θ_a on L by $(x, y) \in \theta_a$ iff $x \vee a = y \vee a$.

Proposition 3 *For any two elements a and b of a double MS-algebra L, we have*

(1) θ_a *is a lattice congruence on L with Ker* $\theta_a = (a]$,

(2) $a \leq b$ *iff* $\theta_a \subseteq \theta_b$*,*
(3) $a = b$ *iff* $\theta_a = \theta_b$*,*
(4) $\theta_0 = \Delta$ *and* $\theta_1 = \nabla$*,*
(5) θ_a *is the smallest lattice congruence containing* $(0, a)$*.*

Proof (1). Obviously θ_a is an equivalence relation on L. Let $(x, y) \in \theta_a$. Then, $x \vee a = y \vee a$. For all $z \in L$, by associativity and commutativity of $\vee$, we have

$$\begin{aligned}(x \vee z) \vee a &= x \vee (z \vee a)\\ &= x \vee (a \vee z)\\ &= (x \vee a) \vee z\\ &= y \vee (a \vee z\\ &= y \vee (z \vee a)\\ &= (y \vee z) \vee a,\end{aligned}$$

and

$$\begin{aligned}(x \wedge z) \vee a &= (x \vee a) \wedge (z \vee a) \text{ by distributivity of } L\\ &= (y \vee a) \wedge (z \vee a)\\ &= (y \wedge z) \vee a \text{ by distributivity of } L.\end{aligned}$$

Then, by Theorem 2, θ_a is a lattice congruence on L. Now

$$\begin{aligned}Ker\, \theta_a &= \{x \in L : (0, x) \in \theta_a\}\\ &= \{x \in L : a = 0 \vee a = x \vee a\}\\ &= \{x \in L : x \leq a\} =[a).\end{aligned}$$

(2) Let $a \leq b$ and $(x, y) \in \theta_a$. Hence, $x \vee a = y \vee a$. Then, $x \vee a \vee b = y \vee a \vee b$ implies $x \vee b = y \vee b$. This gives $(x, y) \in \theta_a$ and $\theta_a \subseteq \theta_b$. Conversely, let $\theta_a \subseteq \theta_b$. Since $(a \wedge b) \vee a = a = a \vee a$, then $(a \wedge b, a) \in \theta_a$. By hypotheses, $(a \wedge b, a) \in \theta_b$. Then, $(a \wedge b) \vee b = a \vee b$ implies $b = a \vee b$. Therefore, $a \leq b$.
(3) It is obvious.
(4) Since for any $(x, y) \in \theta_0$, we have $x = y$. Then, $\theta_0 = \Delta$. For all $x, y \in L$, we have $x \vee 1 = 1 = y \vee 1$ and hence $(x, y) \in \theta_1$. Hence, $\theta_1 = \nabla$.
(5) Let θ be a lattice congruence containing $(0, a)$. Suppose that $(x, y) \in \theta_a$. Then, $x \vee a = y \vee a$. Since $(x, x), (0, a) \in \theta$, then $(x, x \vee a) \in \theta$. Also, $(y, y), (0, a) \in \theta$ give $(y, y \vee a) \in \theta$. Then, $(x, x \vee a), (x \vee a, y) \in \theta$ imply $(x, y) \in \theta$. So, $\theta_a \subseteq \theta$. □

Proposition 4 *For any two elements a and b of a double MS-algebra L, we have*
(1) $\theta_{a\wedge b} = \theta_a \cap \theta_b$*,*
(2) $\theta_{a\vee b} = \theta_a \vee \theta_b$*,*
(3) $\theta_a \circ \theta_b = \theta_b \circ \theta_a$*,*
(4) $\theta_a \circ \theta_b = \theta_a \vee \theta_b$*,*

Proof (1). Since $a \wedge b \leq a, b$, then by Proposition 3(2), $\theta_{a\wedge b} \subseteq \theta_a, \theta_b$. Thus, $\theta_{a\wedge b} \subseteq \theta_a \cap \theta_b$. Conversely, let $(x, y) \in \theta_a \cap \theta_b$. Then

$$\begin{aligned}(x,y) \in \theta_a \cap \theta_b &\Rightarrow (x,y) \in \theta_a \text{ and } (x,y) \in \theta_b \\ &\Rightarrow x \vee a = y \vee a \text{ and } x \vee b = y \vee b \\ &\Rightarrow (x \vee a) \wedge (x \vee b) = (y \vee a) \wedge (y \vee b) \\ &\Rightarrow x \vee (a \wedge b) = y \vee (a \wedge b) \text{ by distributivity of } L \\ &\Rightarrow (x,y) \in \theta_{a\wedge b}.\end{aligned}$$

Therefore, $\theta_a \cap \theta_b \subseteq \theta_{a\wedge b}$ and $\theta_{a\wedge b} = \theta_a \cap \theta_b$.
(2) Since $a, b \leq a \vee b$, then $\theta_a, \theta_b \subseteq \theta_{a\vee b}$. Hence, $\theta_{a\vee b}$ is an upper bound of θ_a and θ_b. Assume that θ_c is an upper bound of θ_a and θ_b. Then, by Proposition 3(2), $\theta_a, \theta_b \subseteq \theta_c$ imply that $a, b \leq c$. We prove that $\theta_{a\vee b} \subseteq \theta_c$. Let $(x, y) \in \theta_{a\vee b}$. Then, $x \vee a \vee b = y \vee a \vee b$. Hence, $x \vee a \vee b \vee c = y \vee a \vee b \vee c$ implies $x \vee c = y \vee c$ and $(x, y) \in \theta_c$. This shows that $\theta_{a\vee b}$ is the least upper bound of θ_a and θ_b, that is, $\theta_{a\vee b} = \theta_a \vee \theta_b$.
(3) Let $(x, y) \in \theta_a \circ \theta_b$. Then, there exists $q \in L$ such that $(x, q) \in \theta_a$ and $(q, y) \in \theta_b$. Thus, $x \vee a = q \vee a$ and $q \vee b = y \vee b$. Put $s = (a \vee y) \wedge (b \vee x)$. Now

$$\begin{aligned}a \vee s &= a \vee \{(a \vee y) \wedge (b \vee x)\} \\ &= (a \vee a \vee y) \wedge (a \vee b \vee x) \text{ by distributivity of } L \\ &= (a \vee y) \wedge (a \vee b \vee q) \text{ as } a \vee q = a \vee x \\ &= (a \vee y) \wedge (a \vee b \vee y) \text{ as } b \vee q = b \vee y \\ &= a \vee \{y \wedge (b \vee y)\} \text{ by distributivity of } L \\ &= a \vee y \text{ by the absorbtion identity.}\end{aligned}$$

Then, $(s, y) \in \theta_a$. Also

$$\begin{aligned}b \vee s &= b \vee \{(a \vee y) \wedge (b \vee x)\} \\ &= (a \vee b \vee y) \wedge (b \vee x) \text{ by distributivity of } L \\ &= (b \vee a \vee q) \wedge (b \vee x) \text{ as } b \vee q = b \vee y \\ &= (b \vee a \vee x) \wedge (b \vee x) \text{ as } x \vee a = q \vee a \\ &= b \vee \{(a \vee x) \wedge x\} \text{ by distributivity of } L \\ &= b \vee x \text{ by the absorbtion identity.}\end{aligned}$$

Then, $(x, s) \in \theta_b$. Therefore, $(x, y) \in \theta_b \circ \theta_a$ and $\theta_a \circ \theta_b \subseteq \theta_b \circ \theta_a$. Conversely, let $(x, y) \in \theta_b \circ \theta_a$. Then, there exists $s \in L$ such that $(x, s) \in \theta_b$ and $(s, y) \in \theta_a$. Set $t = (b \vee y) \wedge (a \vee x)$. Then, we can get $a \vee t = a \vee x$ and $b \vee t = b \vee y$ which means $(x, t) \in \theta_a$ and $(t, y) \in \theta_b$. Therefore, $(x, y) \in \theta_a \circ \theta_b$. So, $\theta_b \circ \theta_a \subseteq \theta_a \circ \theta_b$.
(4) Let $(x, y) \in \theta_a \circ \theta_b$. Then, there exists $q \in L$ such that $(x, q) \in \theta_a$ and $(q, y) \in \theta_b$. Then, $x \vee a = q \vee a$ and $q \vee b = y \vee b$. Using associativity and commutativity of $\vee$, we get

$$(a \vee b) \vee x = (a \vee x) \vee b = (a \vee q) \vee b = a \vee (q \vee b) = a \vee (y \vee b) = (a \vee b) \vee y.$$

Then, $(x, y) \in \theta_{a\vee b}$. Conversely, let $(x, y) \in \theta_{a\vee b}$. Then, $a \vee b \vee x = a \vee b \vee y$. Set $q = (a \vee x) \wedge (b \vee y)$. We have

$$\begin{aligned} a \vee q &= a \vee \{(a \vee x) \wedge (b \vee y)\} \\ &= (a \vee x) \wedge (a \vee b \vee y) \text{ by distributivity of } L \\ &= (a \vee x) \wedge (a \vee b \vee x) \\ &= a \vee x \text{ as } a \vee x \leq a \vee b \vee x. \end{aligned}$$

Then, $(x, q) \in \theta_a$. Also, we can get $(q, y) \in \theta_b$. Therefore, $(x, y) \in \theta_a \circ \theta_b$ and $\theta_{a\vee b} \subseteq \theta_a \circ \theta_b$. □

Theorem 9 *For any two elements a and b of a double MS-algebra L, we have*
(1) θ_a is compatible with $^\circ$ if and only if $a \vee a^\circ = 1$,
(2) θ_a is compatible with $^+$ if and only if $a^+ \wedge a^{++} = 0$,
(3) θ_a is a congruence on L if and only if $a \in C(L)$.

Proof (1). Let $(x, y) \in \theta_a$ and $a^\circ \vee a = 1$. Then, $x \vee a = y \vee a$. We prove that $(x, y) \in \theta_a$ implies $(x^\circ, y^\circ) \in \theta_a$.

$$\begin{aligned} (x, y) \in \theta_a &\Rightarrow x \vee a = y \vee a \\ &\Rightarrow x^\circ \wedge a^\circ = (x \vee a)^\circ = (y \vee a)^\circ = y^\circ \wedge a^\circ \text{ by Theorem 1(5)} \\ &\Rightarrow (x^\circ \wedge a^\circ) \vee a = (y^\circ \wedge a^\circ) \vee a \text{ by joining two sides with } a \\ &\Rightarrow (x^\circ \vee a) \wedge (a^\circ \vee a) = (x^\circ \vee a) \wedge (a^\circ \vee a) \text{ by the distributivity of } L \\ &\Rightarrow x^\circ \vee a = x^\circ \vee a \text{ as } a^\circ \vee a = 1 \\ &\Rightarrow (x^\circ, y^\circ) \in \theta_a \end{aligned}$$

Then, $(x^\circ, y^\circ) \in \theta_a$. Conversely, let θ_a is compatible with $^\circ$. Since $(0, a) \in \theta_a$ by Proposition 3(5), then $(1, a^\circ)) \in \theta_a$. Hence, $(a, a), (1, a^\circ)) \in \theta_a$ implies $(1, a \vee a^\circ) \in \theta_a$. Therefore, $1 = 1 \vee a = a \vee (a \vee a^\circ) = a \vee b$.
(2) Let $a^+ \wedge a^{++} = 0$. Using the properties of dual *MS*-algebra $(L;^+)$ and Proposition 1, we get $a^+ \vee a \geq a^+ \vee a^{++} = (a^+ \wedge a^{++})^+ = 0^+ = 1$ and hence $a^+ \vee a = 1$. Now, let $(x, y) \in \theta_a$. We have

$$\begin{aligned} (x, y) \in \theta_a &\Rightarrow x \vee a = y \vee a \\ &\Rightarrow x^+ \wedge a^+ = (x \vee a)^+ = (y \vee a)^+ = y^+ \wedge a^+ \text{ by Proposition 1(5)} \\ &\Rightarrow (x^+ \wedge a^+) \vee a = (y^+ \wedge a^+) \vee a \text{ by joining two sides with } a \\ &\Rightarrow (x^+ \vee a) \wedge (a^+ \vee a) = (x^+ \vee a) \wedge (a^+ \vee a) \text{ by the distributivity of } L \\ &\Rightarrow x^+ \vee a = x^+ \vee a \text{ as } a^+ \vee a = 1. \end{aligned}$$

Then, $(x^+, y^+) \in \theta_a$. Conversely, let θ_a is compatible with $^+$. Then, $(0, a) \in \theta_a$ implies $(1, a^+)) \in \theta_a$. Since $(a, a), (1, a^+)) \in \theta_a$, then $(1, a \vee a^+) \in \theta_a$. Hence, $1 = 1 \vee a = a \vee a^+$. It follows that $a^+ \wedge a^{++} = (a \vee a^+)^+ = 1^+ = 0$.
(3) As $a \in C(L)$, then $a \vee a^\circ = 1$, $a \wedge a^+ = 0$, and $a = a^{\circ\circ}$, the proof follows (1) and (2). □

Now, we introduce the concept of factor congruences for double *MS*-algebras.

Definition 6 *A congruence θ on a double MS-algebra L is called a factor congruence if there is a congruence ψ on L such that $\theta \wedge \psi = \Delta$, $\theta \vee \psi = \nabla$ and θ permutes with ψ.*

Theorem 10 *Let L be a double MS-algebra and θ a congruence on L. Then, θ is a factor congruence on L if and only if $\theta = \theta_a$ for some $a \in C(L)$.*

Proof Let $a \in C(L)$. Hence, $a^\circ \in C(L)$. Using Theorem 9(3), we deduce that θ_a and θ_{a° are congruences on L. Hence, we get

$$\begin{aligned}
\theta_a \vee \theta_{a^\circ} &= \theta_{a\vee a^\circ} \text{ by Proposition 4(2)}\\
&= \theta_1 \text{ as } a \vee a^\circ = 1\\
&= \nabla \text{ by Proposition 3(4)},\\
\theta_a \cap \theta_{a^\circ} &= \theta_{a\wedge a^\circ} \text{ by Proposition 4(1)}\\
&= \theta_0 \text{ as } a \wedge a^\circ = 0\\
&= \Delta \text{ by Proposition 3(4)},\\
\theta_a \circ \theta_{a^\circ} &= \theta_{a^\circ} \circ \theta_a \text{ by Proposition 4(3)}.
\end{aligned}$$

Therefore, θ_a is a factor congruence on L, whenever $a \in C(L)$. Conversely, let θ be a factor congruence on L. Then, there exists a congruence ψ on L such that $\theta \vee \psi = \nabla$ and $\theta \cap \psi = \Delta$. Since $(0, 1) \in \nabla = \theta \vee \psi = \theta \circ \psi$, then there exists $x \in L$ such that $(0, x) \in \theta$ and $(x, 1) \in \psi$. Thus, $(0, x^{\circ\circ}) \in \theta$ and $(x^{\circ\circ}, 1) \in \psi$. We prove that $\theta = \theta_{x^{\circ\circ}}$ such that $x^{\circ\circ} \in C(L)$. Since $(0, x^{\circ\circ}) \in \theta$, then by Proposition 3(5), $\theta_{x^{\circ\circ}} \subseteq \theta$. Now, let $(p, q) \in \theta$. Then, $(p, q), (x^{\circ\circ}, x^{\circ\circ}) \in \theta$ implies $(p \vee x^{\circ\circ}, q \vee x^{\circ\circ}) \in \theta$. Since $(x^{\circ\circ}, 1), (p, p), (q, q) \in \psi$, then $(x^{\circ\circ} \vee p, 1), (x^{\circ\circ} \vee q, 1) \in \psi$. Hence, $(x^{\circ\circ} \vee p, x^{\circ\circ} \vee q) \in \psi$. Therefore, $(x^{\circ\circ} \vee p, x^{\circ\circ} \vee q) \in \theta \cap \psi = \Delta$. It follows that $x^{\circ\circ} \vee p = x^{\circ\circ} \vee q$ and hence $(p, q) \in \theta_{x^{\circ\circ}}$. So, $\theta \subseteq \theta_{\circ\circ}$ and $\theta = \theta_{x^{\circ\circ}}$. This deduce that $\theta_{x^{\circ\circ}}$ is a congruence on L. So, by Theorem 9(3), $x^{\circ\circ} \in C(L)$. □

Now, we introduce the concept of balanced factor congruences of a double *MS*-algebra.

Definition 7 *A congruence θ on a double MS-algebra L is called balanced if $(\theta \vee \alpha) \cap (\theta \vee \acute{\alpha}) = \theta$ for all factor congruence α and its complement $\acute{\alpha}$. The set $\mathbf{B}(L)$ of all balanced factor congruences which admit a balanced complement is called the Boolean center of L.*

Example 2 *Consider the double MS-algebra L as in Example* 1*. Factor congruences on L are given as follows:*

$$\theta_0 = \Delta, \theta_1 = \nabla, \theta_a = \{\{0, c, a\}, \{x, y, z\}, \{b, d, 1\}\}, \theta_b = \{\{0, x, b\}, \{c, y, d\}, \{a, z, 1\}\}.$$

It is observed that the Boolean lattice $\mathbf{B}(L)$, of all balanced factor congruences is $\mathbf{B}(L) = \{\theta_0, \theta_a, \theta_b, \theta_1\}$ which is represented in Fig. 5. Clearly $C(L)$ and $\mathbf{B}(L)$ are isomorphic Boolean lattices.

Lemma 1 *Let L be a double MS-algebra and $x \in C(L)$. Then, θ_x is balanced.*

Proof Let α be a factor congruence on L and $\acute{\alpha}$ be its complement. Using Theorem 10, there exist $a, b \in C(L)$ such that $\alpha = \theta_a$ and $\acute{\alpha} = \theta_b$. Hence, $\alpha \cap \acute{\alpha} = \Delta$ and $\alpha \vee \acute{\alpha} = \nabla$.

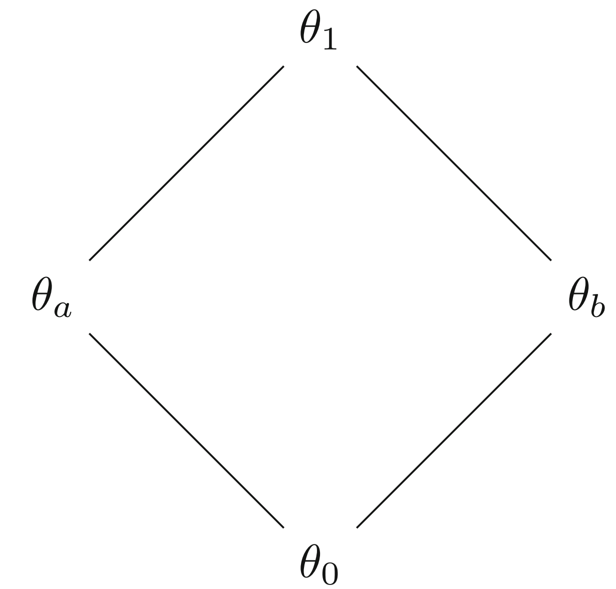

Fig. 5 $\mathbf{B}(L)$

We have

$$\begin{aligned}(\theta_x \vee \alpha) \cap (\theta_x \vee \acute{\alpha}) &= (\theta_x \vee \theta_a) \cap (\theta_x \vee \theta_b)\\ &= \theta_{x\vee a} \cap \theta_{x\vee b} \text{ by Proposition 4(2)}\\ &= \theta_{(x\vee a)\wedge(x\vee b)} \text{ by Proposition 4(1)}\\ &= \theta_{x\vee(a\wedge b)} \text{ by distributivity of } L\\ &= \theta_x \vee (\theta_a \cap \theta_b) \text{ by Proposition 4(2) and (1), respectively}\\ &= \theta_x \vee (\alpha \cap \acute{\alpha})\\ &= \theta_x \vee \Delta \text{ as } \alpha \cap \acute{\alpha} = \Delta\\ &= \theta_x \text{ as } \Delta \subseteq \theta_x \text{for all } x \in L.\end{aligned}$$

Then, θ_x is balanced. □

We close this section with the following two important results.

Theorem 11 *Let L be a double MS-algebra. Then, the Boolean center $\mathbf{B}(L)$ of L is precisely the set $\{\theta_a : a \in C(L)\}$.*

Theorem 12 *Let L be a double MS-algebra. Then, the Boolean center $\mathbf{B}(L)$ is a Boolean algebra and the mapping $a \mapsto \theta_a$ is an isomorphism of $C(L)$ onto $\mathbf{B}(L)$.*

Proof The set of all balanced factor congruences of L is $\mathbf{B}(L) = \{\theta_a : a \in C(L)\}$ by Theorem 11. It is clear that $\theta_1 = \nabla$ is the greatest element of $\mathbf{B}(L)$ and $\theta_0 = \Delta$ is the smallest element of $\mathbf{B}(L)$ by Proposition 3(4). Also, by Proposition 4(1),(2), respectively, we have $\theta_a \cap \theta_b = \theta_{a\wedge b}$ and $\theta_a \vee \theta_b = \theta_{a\vee b}$ for all $\theta_a, \theta_b \in \mathbf{B}(L)$. Then, $(\mathbf{B}(L); \cap, \vee, \theta_0, \theta_1)$ is a bounded lattice. For all $\theta_a, \theta_b, \theta_c \in \mathbf{B}(L)$, by distributivity of $C(L)$, we get $\theta_a \cap (\theta_b \vee \theta_c) = \theta_a \cap \theta_{b\vee c} = \theta_{a\wedge(a\vee b)} = \theta_{(a\vee b)\wedge(a\vee c)} = \theta_{a\vee b} \cap \theta_{a\vee c} = (\theta_a \vee \theta_b) \cap (\theta_a \vee \theta_c)$. Thus, $\mathbf{B}(L)$ is a distributive lattice. The complement of θ_a is θ_{a°. Then, $\mathbf{B}(L)$ is a Boolean algebra. The proof of the rest part of this theorem is straightforward. $\square$

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