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Bi-univalent properties for certain class of Bazilevič functions defined by convolution and with bounded boundary rotation

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Abstract

In this paper, we obtain bi-univalent properties for certain class of Bazilevič functions defined by convolution and with bounded boundary rotation. We will find coefficient bounds for $|a_2|$ and $|a_3|$ for the class $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$.

Keywords: Bi-univalent, Bazilevič functions, Hadamard product, Bounded boundary rotation

2010 Mathematics Subject Classification: 30C45, 30C50

Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad (z\in\mathbb{U}:\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}). \tag{1}$$

For $h(z)\in\mathcal{A}$, given by $h(z)=z+\sum_{n=2}^{\infty}h_nz^n$, the Hadamard product (or convolution) of $f(z)$ and $h(z)$ is defined by:

$$(f*h)(z)=z+\sum_{n=2}^{\infty}a_nh_nz^n=(h\times f)(z). \tag{2}$$

Definition 1 ([1, 2], and [3] **with p = 1**). *Let $\mathcal{P}_k^{\lambda}(\rho)$ $(0\le\rho<1,\ k\ge 2$ and $|\lambda|<\frac{\pi}{2})$ denote the class of functions $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$, which are analytic in $\mathbb{U}$ and satisfy the conditions:*

(*i*) $p(0)=1$,

$$(ii)\int_0^{2\pi}\left|\frac{\Re\left\{e^{i\lambda}p(z)\right\}-\rho\cos\lambda}{1-\rho}\right|\le k\pi\cos\lambda\ \left(r<1,z=re^{i\theta}\in\mathbb{U}\right). \tag{3}$$

We note that:

(i) $\mathcal{P}_k^{\lambda}(0)=\mathcal{P}_k^{\lambda}$ $(k\ge 2$ and $|\lambda|<\frac{\pi}{2})$ is the class of functions introduced by Robertson (see [4]), and he derived a variational formula for functions in this class.

(ii) $\mathcal{P}_k^0(\rho) = \mathcal{P}_k(\rho)$ $(0 \le \rho < 1,\ k \ge 2)$ is the class of functions introduced by Padmanabhan and Parvatham [5] (see also Umarani and Aouf [6]).

(iii) $\mathcal{P}_k^0(0) = \mathcal{P}_k (k \ge 2)$ is the class of functions having their real parts bounded in the mean on $\mathbb{U}$, introduced by Robertson [4] and studied by Pinchuk [7].

(iv) $\mathcal{P}_2^0(\rho) = \mathcal{P}(\rho)$ $(0 \le \rho < 1)$ is the class of functions with positive real part of order ρ, $0 \le \rho < 1$.

(v) $\mathcal{P}_2^0(0) = \mathcal{P}$ is the class of functions having positive real part for $z \in \mathbb{U}$.

By the Koebe one-quarter theorem [8], we know that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains the disk with center in the origin and radius 1/4. Therefore, every univalent function f has an inverse f^{-1} satisfies:

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w \ (|w| < r_0(f),\ r_0(f) \ge 1/4). \tag{4}$$

It is easy to see that the inverse function has the form:

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + ... \ . \tag{5}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and its inverse map $g = f^{-1}$are univalent in $\mathbb{U}$.

Let $\sum$ denote the class of bi-univalent functions in $\mathbb{U}$ in the form (1). For interesting examples about the class $\sum$, see [9].

The object of this paper is to introduce new subclass of Bazilevič functions [10] for the class $\sum$ with bounded boundary rotation and defined by using convolution as follows:

Definition 2 *Let $f, h \in \sum$, $\alpha \in \mathbb{C}^*$, $\beta \ge 0$, $0 \le \rho < 1$, $k \ge 2$ and $|\lambda| < \frac{\pi}{2}$, then $(f*h)(z) \in \sum$ is said to be in the class $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$ if it satisfies the following conditions:*

$$\left\{(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta} + \alpha \frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\right\} \in \mathcal{P}_k^{\lambda}(\rho) \ (z \in \mathbb{U}) \tag{6}$$

and

$$\left\{(1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta} + \alpha \frac{w((f*h)^{-1}(w))'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\right\} \in \mathcal{P}_k^{\lambda}(\rho) \ (w \in \mathbb{U}). \tag{7}$$

We note that by putting different values for h, α, β, k, λ, and ρ, in the above definition, we have:

(1) $\mathcal{M}_{1,0,\rho,k,\beta}\left(f \times \frac{z}{1-z}\right) = R_{\sum}(\rho,k,\beta)$ $(f \in \sum,\ \beta \ge 0,\ 0 \le \rho < 1,\ k \ge 2)$ (see [11], with $\gamma = 1$);

(2) $\mathcal{M}_{\alpha,0,\rho,k,1}(f*h) = \mathcal{L}_{\alpha,\rho,k}(f*h)$ $\left(f,h \in \sum,\ \alpha \in \mathbb{C}^*,\ 0 \le \rho < 1,\ k \ge 2\right)$ (see [12]);

(3) $\mathcal{M}_{\eta,0,\rho,2,1}(f*h) = \mathcal{L}_{\eta,\rho}(f*h)$ $\left(f,h \in \sum,\ \eta \ge 0,\ 0 \le \rho < 1\right)$ (see [13] and [14]);

(4) $\mathcal{M}_{\eta,0,\rho,2,1}\left(f \times \frac{z}{1-z}\right) = \mathcal{L}_{\eta,\rho}(f)(z)$ $\left(f \in \sum,\ \eta \ge 0,\ 0 \le \rho < 1\right)$ (see [15]);

(5) $\mathcal{M}_{1,0,\rho,2,\beta}\left(f \times \frac{z}{1-z}\right) = \mathcal{L}_{\rho,\beta}(f)(z)$ $\left(f \in \sum,\ \beta \ge 0,\ 0 \le \rho < 1\right)$ (see [16]);

(6) $\mathcal{M}_{1,0,\rho,2,1}\left(f \times \frac{z}{1-z}\right) = \mathcal{L}_{\rho}(f)(z)$ $\left(f \in \sum,\ 0 \le \rho < 1\right)$ (see [9]);

(7) $\mathcal{M}_{\alpha,0,\rho,2,\beta}\left(f\times\frac{z}{1-z}\right)=\mathcal{NP}_{\sum}^{\beta,\alpha}(0,\rho)\ \left(f\in\sum,\ \beta,\alpha\geq 0,\ 0\leq\rho<1\right)$ (see [[17], with $\beta=0$]);

(8) $\mathcal{M}_{1,0,\rho,2,\beta}\left(f\times\frac{z}{1-z}\right)=\mathcal{R}_{\sum}(\beta,\rho)\ \left(f\in\sum,\ \beta\geq 0,\ 0\leq\rho<1\right)$ (see [18]).

Also, we can obtain the following subclasses:

(i) $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}\left(f\times\frac{z}{1-z}\right)=F_{\alpha,\lambda,\rho,k,\beta}(f)$

$$=\left\{f\in\sum:(1-\alpha)\left(\frac{f(z)}{z}\right)^{\beta}+\alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\beta}\in\mathcal{P}_k^{\lambda}(\rho)\right.$$
$$\left.\text{and }\ (1-\alpha)\left(\frac{f^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w\left((f^{-1}(w)\right)'}{f^{-1}(w)}\left(\frac{f^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_k^{\lambda}(\rho)\right\};$$

(ii) $\mathcal{M}_{\alpha,0,\rho,k,\beta}(f*h)=\mathcal{F}_{\alpha,\rho,k,\beta}(f*h)$

$$=\left\{f,h\in\sum:(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta}+\alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\in\mathcal{P}_k(\rho)\right.$$
$$\left.\text{and }\ (1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w((f*h)^{-1}(w))'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_k(\rho)\right\};$$

(iii) $\mathcal{M}_{\alpha,0,\rho,2,\beta}(f*h)=\mathcal{F}_{\alpha,\rho,\beta}(f*h)$

$$=\left\{f,h\in\sum:\Re\left[(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta}+\alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\right]>\rho\right.$$
$$\left.\text{and }\ \Re\left[(1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w\left((f*h)^{-1}(w)\right)'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\right]>\rho\right\};$$

(iv) $\mathcal{M}_{\alpha,\lambda,0,k,\beta}(f*h)=\mathcal{M}_{\alpha,\lambda,k,\beta}(f*h)$

$$=\left\{f,h\in\sum:(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta}+\alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\in\mathcal{P}_k^{\lambda}\right.$$
$$\left.\text{and }\ (1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w\left((f*h)^{-1}(w)\right)'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_k^{\lambda}\right\};$$

(v) $\mathcal{M}_{\alpha,0,0,k,\beta}(f*h)=\mathcal{M}_{\alpha,k,\beta}(f*h)$

$$=\left\{f,h\in\sum:(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta}+\alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\in\mathcal{P}_k\right.$$
$$\left.\text{and }\ (1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w\left((f*h)^{-1}(w)\right)'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_k\right\};$$

(vi) $\mathcal{M}_{\alpha,0,0,2,\beta}(f*h)=\mathcal{M}_{\alpha,\beta}(f*h)$

$$=\left\{f,h\in\sum:(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta}+\alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\in\mathcal{P}\right.$$
$$\left.\text{and }\ (1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}+\alpha\frac{w((f*h)^{-1}(w))'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}\right\};$$

(vii) $\mathcal{M}_{1,\lambda,\rho,k,\beta}(f*h)=\mathbb{F}_{\lambda,\rho,k,\beta}(f*h)$

$$=\left\{f,h\in\sum:\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\in\mathcal{P}_k^{\lambda}(\rho)\text{ and }\frac{w((f*h)^{-1}(w))'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\in\mathcal{P}_k^{\lambda}(\rho)\right\};$$

or

$$=\left\{f\in\sum:\frac{e^{i\lambda}\left[\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\right]-\rho\cos\lambda-i\sin\lambda}{(1-\rho)\cos\lambda}\in\mathcal{P}_k\right.$$
$$\left.\text{and }\frac{e^{i\lambda}\left[\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\right]-\rho\cos\lambda-i\sin\lambda}{(1-\rho)\cos\lambda}\in\mathcal{P}_k\right\};$$

(viii) $\mathcal{M}_{1,0,\rho,2,\beta}(f*h)=\mathbb{F}_{\rho,\beta}(f*h)$

$$=\left\{f,h\in\sum:\Re\left[\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta}\right]>\rho\right.$$
$$\left.\textit{and }\Re\left[\frac{w((f*h)^{-1}(w))'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta}\right]>\rho\right\}.$$

In order to obtain our main results, we have to recall here the following lemma.

Lemma 1 ([3] **with p = 1**) . *If* $p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}_k^{\lambda}(\rho)$, *then*

$$|c_n|\leq(1-\rho)\,k\,\cos\lambda. \tag{8}$$

The result is sharp. Equality is attained for the odd coefficients and even coefficients respectively for the functions:

$$p_1(z)=1+(1-\rho)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z}{1-z}\right)-1\right],$$
$$p_2(z)=1+(1-\rho)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z^2}{1+z^2}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z^2}{1-z^2}\right)-1\right].$$

We note that for $\lambda=0$ in Lemma 1, we obtain the result obtained by Goswami et al. [19] [Lemma 2.1] for the class $\mathcal{P}_k(\rho)$.

In this paper, we will obtain the coefficients bounds $|a_2|$ and $|a_3|$ for the class $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$, which defined in Definition 2.

Coefficient estimates for functions in the class $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$

Theorem 1 *Let* $f,h\in\sum$, $\alpha\in\mathbb{C}^*\backslash\{-1,\frac{-1}{2}\}$, $\beta\geq0$, $0\leq\rho<1$, $k\geq2$, $|\lambda|<\frac{\pi}{2}$, $f*h$ *given by* (2) *and* $h_2,\ h_3\neq0$. *If* $f*h$ *belongs to* $\mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$, *then:*

$$|a_2|\leq\min\left\{\sqrt{\frac{2k(1-\rho)\cos\lambda}{|2\alpha+\beta|(\beta+1)|h_2|^2}};\ \frac{k(1-\rho)\cos\lambda}{|\alpha+\beta|\,|h_2|}\right\} \tag{9}$$

and

$$|a_3| \le \frac{k(1-\rho)\cos\lambda}{|2\alpha+\beta|\,|h_3|} + \frac{[k(1-\rho)\cos\lambda]^2}{|\alpha+\beta|^2\,|h_3|}. \tag{10}$$

The result is sharp.

Proof 1 *If $(f*h) \in \mathcal{M}_{\alpha,\lambda,\rho,k,\beta}(f*h)$, then from Definition 2, we have:*

$$(1-\alpha)\left(\frac{(f*h)(z)}{z}\right)^{\beta} + \alpha\frac{z(f*h)'(z)}{(f*h)(z)}\left(\frac{(f*h)(z)}{z}\right)^{\beta} = p(z),\ p \in \mathcal{P}_k^{\lambda}(\rho) \tag{11}$$

and

$$(1-\alpha)\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta} + \alpha\frac{w\left((f*h)^{-1}(w)\right)'}{(f*h)^{-1}(w)}\left(\frac{(f*h)^{-1}(w)}{w}\right)^{\beta} = q(w),\ q \in \mathcal{P}_k^{\lambda}(\rho), \tag{12}$$

where p and q have Taylor expansions as follows:

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ..., z \in \mathbb{U}, \tag{13}$$

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ..., w \in \mathbb{U}. \tag{14}$$

By comparing the coefficients in (11) with (13) and coefficients in (12) with (14), we obtain:

$$p_1 = (\beta+\alpha)\ a_2 h_2, \tag{15}$$

$$p_2 = (\beta+2\alpha)\ a_3 h_3 + \frac{(\beta+2\alpha)(\beta-1)}{2}\ a_2^2 h_2^2, \tag{16}$$

$$q_1 = -(\beta+\alpha)\ a_2 h_2 \tag{17}$$

and

$$q_2 = (\beta+2\alpha)\ \left(2a_2^2 h_2^2 - a_3 h_3\right) + \frac{(\beta+2\alpha)(\beta-1)}{2}\ a_2^2 h_2^2. \tag{18}$$

Since $p, q \in \mathcal{P}_k^{\lambda}(\rho)$ and by applying Lemma 1, we have:

$$\left|p_n\right| \le k(1-\rho)\cos\lambda\ (n \ge 1) \tag{19}$$

and

$$\left|q_n\right| \le k(1-\rho)\cos\lambda\ (n \ge 1). \tag{20}$$

From (16) and (18) and using inequalities (19) and (20), we obtain:

$$|a_2|^2 \le \frac{1}{|2\alpha+\beta|\,|\beta+1|}\frac{\left|p_2\right|+\left|q_2\right|}{|h_2|^2} \le \frac{2k(1-\rho)\cos\lambda}{|2\alpha+\beta|\,(\beta+1)\,|h_2|^2}. \tag{21}$$

Also, from (15) and (19), we obtain:

$$|a_2| \le \frac{k(1-\rho)\cos\lambda}{|\alpha+\beta|\,|h_2|}. \tag{22}$$

Subtracting (18) from (16), we have:

$$p_2 - q_2 = 2\,(2\alpha+\beta)\ \left(a_3 h_3 - a_2^2 h_2^2\right). \tag{23}$$

Also, we have:

$$p_1^2 + q_1^2 = 2\,(\alpha+\beta)^2\, a_2^2 h_2^2. \tag{24}$$

After using (23), (24), (19), and (20), and some easily calculations, we obtain:

$$|a_3| \leq \frac{k(1-\rho)\cos\lambda}{|2\alpha+\beta|\,|h_3|} + \frac{[k(1-\rho)\cos\lambda]^2}{|\alpha+\beta|^2\,|h_3|}, \tag{25}$$

which completes the proof of Theorem 1. The result is sharp in view of the fact that assertion (8) of Lemma 1 is sharp.

Remark 1 *For* $h(z)=\frac{z}{1-z}$, $\beta=\alpha=1$, $k=2$, *and* $\lambda=0$ *in Theorem 1, we obtain the result obtained by Srivastava et al. [9] [Theorem 2].*

Putting $h(z)=\frac{z}{1-z}$ in Theorem 1, we obtain the following corollary.

Corollary 1 *Let* $f\in\sum$, $\alpha\in\mathbb{C}^*\backslash\left\{-1,\frac{-1}{2}\right\}$, $\beta\geq 0$, $0\leq\rho<1$, $k\geq 2$ *and* $|\lambda|<\frac{\pi}{2}$. *If* $f\in F_{\alpha,\lambda,\rho,k,\beta}(f)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\rho)\cos\lambda}{|2\alpha+\beta|\,(\beta+1)}};\ \frac{k(1-\rho)\cos\lambda}{|\alpha+\beta|}\right\}$$

and

$$|a_3| \leq \frac{k(1-\rho)\cos\lambda}{|2\alpha+\beta|} + \frac{[k(1-\rho)\cos\lambda]^2}{|\alpha+\beta|^2}.$$

The result is sharp.

Putting $\lambda=0$ in Theorem 1, we obtain the following corollary.

Corollary 2 *Let* $f,h\in\sum$, $\alpha\in\mathbb{C}^*\backslash\left\{-1,\frac{-1}{2}\right\}$, $\beta\geq 0$, $0\leq\rho<1$, $k\geq 2$, $f*h$ *given by (2) and* $h_2,\ h_3\neq 0$. *If* $f*h\in\mathcal{F}_{\alpha,\rho,k,\beta}(f*h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\rho)}{|2\alpha+\beta|\,(\beta+1)\,|h_2|^2}};\ \frac{k(1-\rho)}{|\alpha+\beta|\,|h_2|}\right\}$$

and

$$|a_3| \leq \frac{k(1-\rho)}{|2\alpha+\beta|\,|h_3|} + \frac{[k(1-\rho)]^2}{|\alpha+\beta|^2\,|h_3|}.$$

The result is sharp.

Putting $\lambda=0$ and $k=2$ in Theorem 1, we obtain the following corollary.

Corollary 3 *Let* $f,h\in\sum$, $\alpha\in\mathbb{C}^*\backslash\left\{-1,\frac{-1}{2}\right\}$, $\beta\geq 0$, $0\leq\rho<1$, $f*h$ *given by (2) and* $h_2,\ h_3\neq 0$. *If* $f*h\in\mathcal{F}_{\alpha,\rho,\beta}(f*h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\rho)}{|2\alpha+\beta|\,(\beta+1)\,|h_2|^2}};\ \frac{2(1-\rho)}{|\alpha+\beta|\,|h_2|}\right\}$$

and

$$|a_3| \leq \frac{2(1-\rho)}{|2\alpha+\beta|\,|h_3|} + \frac{[2(1-\rho)]^2}{|\alpha+\beta|^2\,|h_3|}.$$

The result is sharp.

Putting $\alpha=1$ in Theorem 1, we obtain the following corollary.

Corollary 4 *Let* $f, h \in \sum$, $\beta \geq 0$, $0 \leq \rho < 1$, $k \geq 2$, $|\lambda| < \frac{\pi}{2}$, $f * h$ *given by* (2) *and* h_2, $h_3 \neq 0$. *If* $f * h \in \mathbb{F}_{\lambda,\rho,k,\beta}(f * h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\rho)\cos\lambda}{(2+\beta)(\beta+1)|h_2|^2}}; \frac{k(1-\rho)\cos\lambda}{(1+\beta)h_2}\right\}$$

and

$$|a_3| \leq \frac{k(1-\rho)\cos\lambda}{(2+\beta)|h_3|} + \frac{[k(1-\rho)\cos\lambda]^2}{(1+\beta)^2|h_3|}.$$

The result is sharp.

Putting $\alpha = 1$, $k = 2$, and $\lambda = 0$ in Theorem 1, we obtain the following corollary.

Corollary 5 *Let* $f, h \in \sum$, $\beta \geq 0$, $0 \leq \rho < 1$, $f * h$ *given by* (2) *and* h_2, $h_3 \neq 0$. *If* $f * h \in \mathbb{F}_{\rho,\beta}(f * h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{4(1-\rho)}{(2+\beta)(\beta+1)|h_2|^2}}; \frac{2(1-\rho)}{(1+\beta)h_2}\right\}$$

and

$$|a_3| \leq \frac{2(1-\rho)}{(2+\beta)|h_3|} + \frac{[2(1-\rho)]^2}{(1+\beta)^2|h_3|}.$$

The result is sharp.

Putting $\rho = 0$ in Theorem 1, we obtain the following corollary.

Corollary 6 *Let* $f, h \in \sum$, $\alpha \in \mathbb{C}^* \setminus \left\{-1, \frac{-1}{2}\right\}$, $\beta \geq 0$, $|\lambda| < \frac{\pi}{2}$, $k \geq 2$, $f * h$ *given by* (2) *and* h_2, $h_3 \neq 0$. *If* $f * h \in \mathcal{M}_{\alpha,\lambda,k,\beta}(f * h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k\cos\lambda}{|2\alpha+\beta|(\beta+1)|h_2|^2}}; \frac{k\cos\lambda}{|\alpha+\beta||h_2|}\right\}$$

and

$$|a_3| \leq \frac{k\cos\lambda}{|2\alpha+\beta||h_3|} + \frac{[k\cos\lambda]^2}{|\alpha+\beta|^2|h_3|}.$$

The result is sharp.

Putting $\rho = \lambda = 0$ in Theorem 1, we obtain the following corollary.

Corollary 7 *Let* $f, h \in \sum$, $\alpha \in \mathbb{C}^* \setminus \left\{-1, \frac{-1}{2}\right\}$, $\beta \geq 0$, $k \geq 2$, $f * h$ *given by* (2) *and* h_2, $h_3 \neq 0$. *If* $f * h \in \mathcal{M}_{\alpha,k,\beta}(f * h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k}{|2\alpha+\beta|(\beta+1)|h_2|^2}}; \frac{k}{|\alpha+\beta||h_2|}\right\}$$

and

$$|a_3| \leq \frac{k}{|2\alpha+\beta||h_3|} + \frac{k^2}{|\alpha+\beta|^2|h_3|}.$$

The result is sharp.

Putting $\rho = \lambda = 0$ and $k = 2$ in Theorem 1, we obtain the following corollary.

Corollary 8 *Let* $f, h \in \sum$, $\alpha \in \mathbb{C}^{*} \backslash \left\{-1, \frac{-1}{2}\right\}$, $\beta \geq 0, f*h$ *given by* (2) *and* $h_2,\ h_3 \neq 0$. *If* $f*h \in \mathcal{M}_{\alpha,\beta}(f*h)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{4}{|2\alpha+\beta|\,(\beta+1)\,|h_2|^2}};\ \frac{2}{|\alpha+\beta|\,|h_2|}\right\}$$

and

$$|a_3| \leq \frac{2}{|2\alpha+\beta|\,|h_3|} + \frac{4}{|\alpha+\beta|^2\,|h_3|}.$$

The result is sharp.

Putting $\lambda = 0,\ \alpha = 1$ and $h(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the following corollary.

Corollary 9 *Let* $f \in \sum$, $0 \leq \rho < 1$ *and* $\beta \geq 0$. *If* $f \in R_{\sum}(\rho, k, \beta)$, *then:*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\rho)}{(2+\beta)\,(\beta+1)}};\ \frac{k(1-\rho)}{(1+\beta)}\right\}$$

and

$$|a_3| \leq \frac{k(1-\rho)}{(2+\beta)} + \frac{[k(1-\rho)]^2}{(1+\beta)^2}.$$

The result is sharp.

Remark 2 *The results in Corollary 9 correct the results obtained by Orhan et al. [11] [Theorem 2.11, with* $\gamma = 1.$ *].*

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