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Subordination and inclusion theorems for higher order derivatives of a generalized fractional differintegral operator

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Abstract

The main object of this paper is to investigate some subordination results of certain subclasses of multivalent analytic functions which are defined by a generalized fractional differintegral operator. Inclusion relations for functions in the class $\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B)$ and the images of these functions by the generalized Bernardi-Libera-Livingston integral operator are also considered.

Keywords: Differential subordination, Multivalent functions, Higher order derivatives, Generalized fractional differintegral operator

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Introduction

Denote the class consisting of analytic and multivalent functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \ (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

by $\mathcal{A}(p)$. We note that $\mathcal{A}(1) = \mathcal{A}$.

Consider the first-order differential subordination

$$H(\varphi(z), z\varphi'(z)) \prec h(z),$$

where the symbol $\prec$ stands for subordination of two analytic functions in $\mathbb{U}$ (see [1, 2]). A univalent function q is called *dominant*, if $\varphi(z) \prec q(z)$ for all analytic functions φ that satisfy this differential subordination. A dominant $\widetilde{q}$ is called the *best dominant*, if $\widetilde{q}(z) \prec q(z)$ for all dominant q. For $f \in \mathcal{A}(p)$, the qth order derivative of $f(z)$ could be written as

$$f^{(q)}(z) = \delta(p, q) z^{p-q} + \sum_{n=1}^{\infty} \delta(p+n, q) a_{p+n} z^{p+n-q}, \ z \in \mathbb{U} \left(p > q, \ q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right), \tag{2}$$

where

$$\delta(p, q) = \frac{p!}{(p-q)!} := \begin{cases} p(p-1)\dots(p-q+1), & \text{if } q \neq 0, \\ 1, & \text{if } q = 0. \end{cases}$$

Let

$$ {}_pF_q\left(a_1,...,a_p;b_1,...,b_q;z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n...(a_p)_n}{(b_1)_n...(b_q)_n(1)_n}z^n, \tag{3}$$

be the well-known *generalized hypergeometric function* for complex parameters $a_1,...,a_q$, $b_1,...,b_s$ $(b_j\notin\mathbb{Z}_0^-=\{0,-1,-2,...\};\ j=1,2,...,s)$ and $(\lambda)_\nu$ is the Pochhammer symbol defined by

$$ (\lambda)_\upsilon:=\begin{cases}1 & \text{if } \upsilon=0,\\ \lambda(\lambda+1)(\lambda+2)...(\lambda+\upsilon-1) & \text{if } \upsilon\in\mathbb{N}.\end{cases}$$

In addition, if we put $p=2,\ q=1,\ a_1=a,\ a_2=b,\ b_1=c$ in (3), we get the (Gaussian) hypergeometric function ${}_2F_1(a,b;c;z)$ $(c\neq 0,-1,-2,\ldots)$ which satisfies (see [3])

$$ \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt=\tfrac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b;c;z)\ (\Re(c)>\Re(b)>0); \tag{4}$$

$$ {}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right); \tag{5}$$

and

$$ {}_2F_1(a,b;c;z)={}_2F_1(b,a;c;z). \tag{6}$$

We will recall some definitions which will be used in our paper.

Definition 1 [4–12]. *Assume that* $0\leq\lambda<1$ *and* $\mu,\eta\in\mathbb{R}$*. Then, in terms of* ${}_2F_1$*, the generalized fractional derivative operator for* $f\in\mathcal{A}(p)$ *is defined by*

$$ J_{0,z}^{\lambda,\mu,\eta,p}f(z):=\frac{d}{dz}\left[\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z(z-\zeta)^{-\lambda}f(\zeta)\,{}_2F_1\left(\mu-\lambda,1-\eta;1-\lambda;1-\frac{\zeta}{z}\right)d\zeta\right],$$

where f is an analytic function in a simply-connected region of the complex z-plane containing the origin with the order $f(z)=O(|z|^\varepsilon)$, $z\to 0$ *when* $\varepsilon>\max\{0,\mu-\eta\}-1$ *and the multiplicity of* $(z-\zeta)^{-\lambda}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $z-\zeta>0$.

Remark 1 *We note that*
(i) $J_{0,z}^{\lambda,\mu,\eta,p}\left\{z^{p+n}\right\}=\dfrac{\Gamma(p+n+1)\Gamma(p+n+1-\mu+\eta)}{\Gamma(p+n+1-\mu)\Gamma(p+n+1-\lambda+\eta)}z^{p+n-\mu}\ (n\geq 1)$,
(ii) $J_{0,z}^{\lambda,\lambda,\eta,p}f(z)=D_z^\lambda f(z)$ *(see* [13]*).*

Goyal and Prajapat [14] (see also [4–12]) defined the operator $\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}:\mathcal{A}(p)\to\mathcal{A}(p)$ $\left(0\leq\lambda<1,\ \mu<p+1,\ \eta>\max\{\lambda,\mu\}-p-1\right)$, by

$$ \begin{aligned}\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)&:=\frac{\Gamma(p+1-\mu)\Gamma(p+1-\lambda+\eta)}{\Gamma(p+1)\Gamma(p+1-\mu+\eta)}z^\mu J_{0,z}^{\lambda,\mu,\eta,p}f(z)\\ &=z^p+\sum_{n=1}^{\infty}\frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}a_{p+n}z^{p+n}\\ &=z^p\,{}_3F_2\left(1,p+1,p+1-\mu+\eta;p+1-\mu,p+1-\lambda+\eta;z\right)*f(z),\end{aligned} \tag{7}$$

where the symbol $*$ stands for convolution of two power series and $f \in \mathcal{A}(p)$. It is easy to check that

$$z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)' = (p-\mu)\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z) + \mu \mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z). \quad (8)$$

In this paper, we define the higher order derivative of $\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)$ as follows:

$$\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)} = \delta(p,q)\, z^{p-q} + \sum_{n=1}^{\infty} \frac{(p+1)_n (p+1-\mu+\eta)_n}{(p+1-\mu)_n (p+1-\lambda+\eta)_n} \delta(p+n,q)\, a_{p+n} z^{p+n-q}$$
$$\left(p \in \mathbb{N},\ q \in \mathbb{N}_0,\ p > q,\ 0 \le \lambda < 1,\ \mu < p+1,\ \eta > \max\{\lambda,\mu\} - p - 1\right). \quad (9)$$

From (9), we have

$$z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)} = (p-\mu)\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)\right)^{(q-1)} + (\mu - q + 1) \times \left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)} \quad (q \in \mathbb{N}). \quad (10)$$

We say that $f \in \mathcal{A}(p)$ is in the class $\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B)$ if

$$\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}} - \zeta\right) \prec \frac{1+Az}{1+Bz}, \quad (11)$$

$0 \le \lambda < 1,\ \mu < p+1,\ \eta > \max\{\lambda,\mu\} - p - 1,\ 0 \le \zeta < p-q,\ -1 \le B < A \le 1,\ p \in \mathbb{N},\ q \in \mathbb{N}_0$ *and* $p > q + \zeta$. Denoting by $\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta,\xi)$, the class of functions $f(z) \in \mathcal{A}(p)$ which satisfies

$$\Re\left\{\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}} - \zeta\right)\right\} > \xi \ (\xi < 1;\ p \in \mathbb{N};\ z \in \mathbb{U}). \quad (12)$$

Preliminaries

To prove our main results, we shall need the following definition and lemmas.

Definition 2 [2]. *Denote the set of all functions f that are analytic and univalent on* $\overline{\mathbb{U}} \setminus E(f)$ *by* $\mathcal{Q}$, *where*

$$E(f) := \{\varsigma \in \partial\mathbb{U} : \lim_{z\to\varsigma} f(z) = \infty\},$$

and are such that $f'(\varsigma) \neq 0$ *for* $\varsigma \in \partial\mathbb{U} \setminus E(f)$.

Lemma 1 [15]. *Let* $h(z)$ *be analytic and convex (univalent) function in* $\mathbb{U}$ *with* $h(0) = 1$. *Also let* ϕ *given by*

$$\phi(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$$

be analytic in $\mathbb{U}$. *If*

$$\phi(z) + \frac{z\phi'(z)}{\alpha} \prec h(z) \ (\Re(\alpha) \geqslant 0;\ \alpha \neq 0), \quad (13)$$

then

$$\phi(z) \prec \psi(z) = \frac{\alpha}{n} z^{-\frac{\alpha}{n}} \int_0^z t^{\frac{\alpha}{n}-1} h(t)\,dt \prec h(z),$$

and ψ *is the best dominant of (13).*

Lemma 2 [16]. *Let h be a convex functions with*

$$\Re\left[\beta h(z)+\gamma\right]>0 \ \ (\beta,\gamma\in\mathbb{C},\ z\in\mathbb{U}).$$

If $p(z)$ *is analytic in* $\mathbb{U}$ *with* $p(0)=h(0)$*, then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)\Rightarrow p(z)\prec h(z).$$

The class of star-like (and normalized) functions of order α *in* $\mathbb{U}$*, is*

$$\mathcal{S}^*(\alpha)=\left\{f\in\mathcal{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha \ \ (\alpha<1;\ z\in\mathbb{U})\right\}.$$

Also in [17]*, if* $\beta>0$ *and* $\beta+\gamma>0$*, for a given* $\alpha\in\left[-\frac{\gamma}{\beta},1\right)$*, we define the order of star-likeness of the class* $\mathrm{I}_{\beta,\gamma}\left[\mathcal{S}^*(\alpha)\right]$ *by the largest number* $\vartheta(\alpha;\beta,\gamma)$ *such that* $\mathrm{I}_{\beta,\gamma}\left[\mathcal{S}^*(\alpha)\right]\subset\mathcal{S}^*(\vartheta)$*, where* $\mathrm{I}_{\beta,\gamma}$ *is given by*

$$\mathrm{I}_{\beta,\gamma}(f)(z)=\left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt\right]^{\frac{1}{\beta}}. \tag{14}$$

Lemma 3 [17]. *Let* $\beta>0$*,* $\beta+\gamma>0$ *and consider* $\mathrm{I}_{\beta,\gamma}$ *defined by (14). If* $\alpha\in\left[-\frac{\gamma}{\beta},1\right)$*, then the order of starlikeness of the class* $\mathrm{I}_{\beta,\gamma}\left[\mathcal{S}^*(\alpha)\right]$ *is given by the number* $\vartheta(\alpha;\beta,\gamma)=\inf\left\{\Re\left(q(z)\right):z\in\mathbb{U}\right\}$*, where*

$$q(z)=\frac{1}{\beta Q(z)}-\frac{\gamma}{\beta}\ \text{and}\ Q(z)=\int_0^1\left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)}t^{\beta+\gamma-1}dt.$$

Moreover, if $\alpha\in[\alpha_0,1)$*, where* $\alpha_0=\max\left\{\frac{\beta-\gamma-1}{2\beta};-\frac{\gamma}{\beta}\right\}$ *and* $g=\mathrm{I}_{\beta,\gamma}\left(f\right)$ *with* $f\in\mathcal{S}^*(\alpha)$*, then*

$$\Re\left(\frac{zg'(z)}{g(z)}\right)>\vartheta(\alpha;\beta,\gamma)\ \ (z\in\mathbb{U}),$$

where

$$\vartheta(\alpha;\beta,\gamma)=\frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1\left(1,2\beta(1-\alpha),\beta+\gamma+1;\frac{1}{2}\right)}-\gamma\right].$$

Subordination and Inclusion theorems involving $\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q)}$

We assume throughout this paper unless otherwise mentioned that $p\in\mathbb{N}$, $0\leq\lambda<1$, $\mu<p$, $\eta>\max\{\lambda,\mu\}-p-1$, $-1\leq B<A\leq 1$, $0\leq\zeta<p-q$, $\xi<1$, $\sigma>0$, $0<c\leq 1$ and the powers are considered principal ones.

Theorem 1 *Assume that* $1\leq q\leq p$ *and* $f(z)\in\mathcal{A}(p)$ *satisfy*

$$(1-\sigma)\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)}}{\delta(p,q-1)z^{p-q+1}}+\sigma\frac{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q-1)}}{\delta(p,q-1)z^{p-q+1}}\prec\sqrt{1+cz},$$

then

$$\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)}}{\delta(p,q-1)z^{p-q+1}}\prec Q(z)\prec\sqrt{1+cz}, \tag{15}$$

where

$$Q(z) = (1+cz)^{\frac{1}{2}}\, {}_2F_1\left(-\frac{1}{2}, 1; \frac{p-\mu}{\sigma}+1; \frac{cz}{1+cz}\right), \tag{16}$$

is the best dominant of (15). Furthermore,

$$\Re\left\{\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta\,(p,q-1)\, z^{p-q+1}}\right\} > M, \tag{17}$$

where

$$M = (1-c)^{\frac{1}{2}}\, {}_2F_1\left(-\frac{1}{2}, 1; \frac{p-\mu}{\sigma}+1; \frac{c}{c-1}\right).$$

The estimate in (17) is the best possible.

Proof Putting

$$\phi(z) = \frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta\,(p,q-1)\, z^{p-q+1}} \; (z \in \mathbb{U}), \tag{18}$$

then $\phi(z)$ is analytic in $\mathbb{U}$. After some computations, we get

$$(1-\sigma)\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta\,(p,q-1)\, z^{p-q+1}} + \sigma\frac{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)\right)^{(q-1)}}{\delta\,(p,q-1)\, z^{p-q+1}}$$
$$= \phi(z) + \left(\frac{\sigma}{p-\mu}\right) z\phi'(z) \prec \sqrt{1+cz}.$$

where the influence of $h(z) = \sqrt{1+cz}$ under certain values of c is illustrated by Fig. 1. To apply Lemma 1, it suffies to show that $h(z)$ is convex, therefore for $z = re^{i\theta}$, $r \in (0,1)$, $\theta \in [-\pi, \pi]$, we have

$$1 + \frac{zh''}{h'} = 1 - \frac{cz}{2\,(1+cz)} = \frac{2+cz}{2\,(1+cz)},$$

and

$$\Re\left(1 + \frac{zh''}{h'}\right) = \frac{2 + 3cr\cos\theta + c^2r^2}{\left|1+cre^{i\theta}\right|^2} \geq \frac{2 - 3cr + c^2r^2}{\left|1+cre^{i\theta}\right|^2}$$
$$= \frac{(2-cr)\,(1-cr)}{\left|1+cre^{i\theta}\right|^2} > 0.$$

This implies that h is convex in $\mathbb{U}$.

Now, by using Lemma 1 (with $n = 1$) and making a change of variables followed by the use of (4) and (5), we deduce that

$$\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta\,(p,q-1)\, z^{p-q+1}} \prec Q(z) = \frac{p-\mu}{\sigma} z^{-\frac{p-\mu}{\sigma}} \int_0^z t^{\frac{p-\mu}{\sigma}-1} (1+ct)^{\frac{1}{2}}\, dt$$
$$= (1+cz)^{\frac{1}{2}}\, {}_2F_1\left(-\frac{1}{2}, 1; \frac{p-\mu}{\sigma}+1; \frac{cz}{1+cz}\right),$$

this proves (15). Next, it is enough to show that

$$\inf_{|z|<1} \{\Re(Q(z)\} = Q(-1).$$

Indeed

$$\Re\left\{\sqrt{1+cz}\right\} \geqslant \sqrt{1-cr} \;\; (|z| \leq r < 1).$$

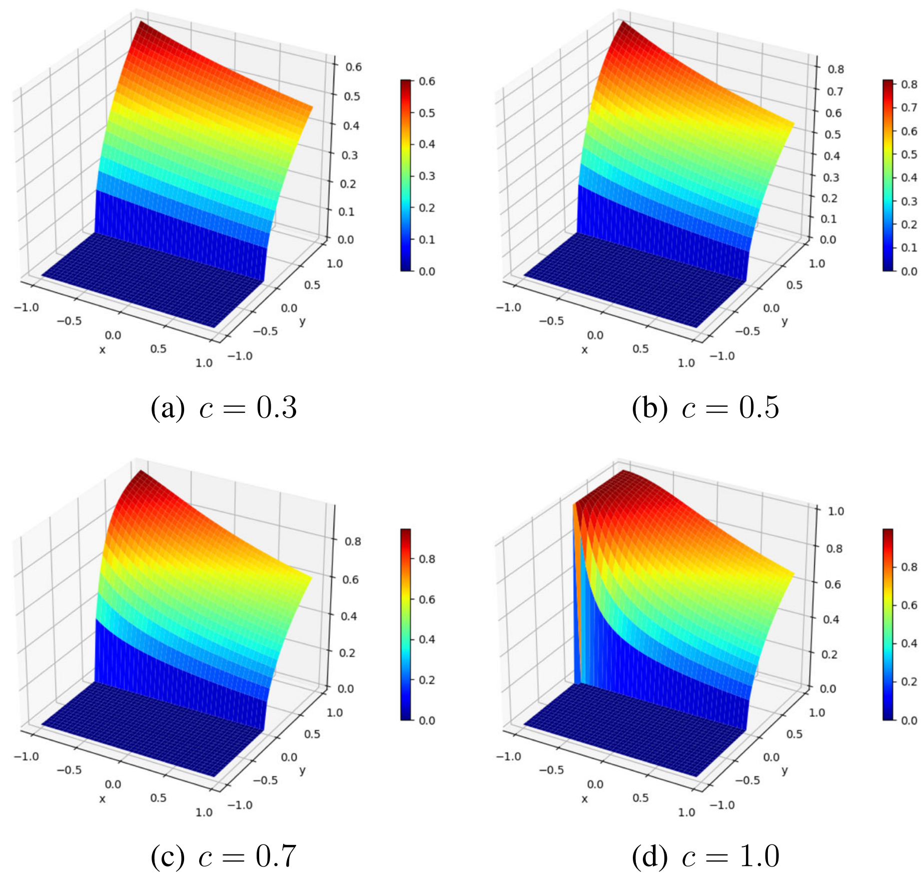

Fig. 1 Influence of $h(z) = \sqrt{1 + cz}$ for different values of c

Setting

$$G(z,s) = \sqrt{1 + czs} \text{ and } d\nu(s) = \frac{p-\mu}{\sigma} s^{\frac{p-\mu}{\sigma}-1} ds \ (0 \leq s \leq 1),$$

which is a positive measure on the closed interval $[0, 1]$, we get

$$Q(z) = \int_0^1 G(z,s) d\nu(s),$$

so that

$$\Re \{Q(z)\} \geqslant \int_0^1 \sqrt{1 - cr} d\nu(s) = Q(-r) \ (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain (17). To show that the result in (17) is sharp, let us suppose that

$$\Re \left\{ \frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta(p, q-1) z^{p-q+1}} \right\} > M_1,$$

that is

$$\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q-1)}}{\delta(p, q-1) z^{p-q+1}} \prec \frac{1 + (1 - 2M_1) z}{1 - z}.$$

From (15), we have

$$\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)}}{\delta\left(p,q-1\right)z^{p-q+1}} \prec \frac{1+(1-2M)\,z}{1-z},$$

and so

$$\frac{1+(1-2M)\,z}{1-z} \prec \frac{1+(1-2M_1)\,z}{1-z},$$

which implies that $M \leq M_1$, that is, M cannot be decreased and the estimate in (17) is the best possible. □

For $f \in \mathcal{A}(p)$ the generalized Bernardi-Libera-Livingston integeral operator $F_{p,\upsilon}$ is defined by (see [18]):

$$\begin{aligned} F_{p,\upsilon}f(z) &= \frac{\upsilon+p}{z^p}\int_0^z t^{\upsilon-1}f(t)dt \\ &= \left(z^p + \sum_{n=1}^{\infty}\frac{\upsilon+p}{\upsilon+p+n}z^{p+n}\right) * f(z) \\ &= z^p\,{}_3F_2\left(1,1,\upsilon+p;1,\upsilon+p+1;z\right) * f(z)\ (\upsilon > -p). \end{aligned} \tag{19}$$

Lemma 4 *If* $f \in \mathcal{A}\left(p\right)$, *then (i)* $\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}\left(F_{p,\upsilon}f\right) = F_{p,\upsilon}\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f\right)$,
(ii)

$$z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)' = (p+\upsilon)\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z) - \upsilon\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z), \tag{20}$$

(iii)

$$z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q)} = (p+\upsilon)\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)} - (\upsilon+q-1)\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}. \tag{21}$$

Proof Since $f(z) \in \mathcal{A}(p)$, then

$$\begin{aligned} \mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}\left(F_{p,\upsilon}f\right) &= \left[z^p{}_3F_2\left(1,p+1,p+1-\mu+\eta;p+1-\mu,p+1-\lambda+\eta;z\right)\right] * \left(F_{p,\upsilon}f\right) \\ &= \left[z^p{}_3F_2\left(1,p+1,p+1-\mu+\eta;p+1-\mu,p+1-\lambda+\eta;z\right)\right] \\ &\quad * \left[z^p{}_3F_2\left(1,1,\upsilon+p;1,\upsilon+p+1;z\right) * f(z)\right], \end{aligned}$$

and

$$\begin{aligned} F_{p,\upsilon}\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f\right) &= z^p{}_3F_2\left(1,1,\upsilon+p;1,\upsilon+p+1;z\right) * \left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f\right) \\ &= z^p{}_3F_2\left(1,1,\upsilon+p;1,\upsilon+p+1;z\right) * \\ &\quad \left[z^p{}_3F_2\left(1,p+1,p+1-\mu+\eta;p+1-\mu,p+1-\lambda+\eta;z\right) * f(z)\right]. \end{aligned}$$

Now, the first part of this lemma follows. Also, the recurrence relation of $F_{p,\upsilon}$ is given by

$$z\left(F_{p,\upsilon}f(z)\right)' = (p+\upsilon)f(z) - \upsilon F_{p,\upsilon}f(z). \tag{22}$$

If we replace $f(z)$ by $\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)$ and using the first part of this lemma, we get (20). If we differentiate (20) q-times, we obtain (21). □

Theorem 2 *Suppose that* $1 \leq q \leq p$ *and* $f(z) \in \mathcal{A}(p)$ *satisfy*

$$(1-\sigma)\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}+\sigma\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}\prec\sqrt{1+cz},$$

where $F_{p,\upsilon}$ *defined by (19), then*

$$\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}\prec\varphi(z)\prec\sqrt{1+cz}, \tag{23}$$

where $\varphi(z)$ *given by*

$$\varphi(z)=(1+cz)^{\frac{1}{2}}\,{}_2F_1\left(-\frac{1}{2},1;\frac{\upsilon+p}{\sigma}+1;\frac{cz}{1+cz}\right),$$

is the best dominant of (23). Further,

$$\Re\left\{\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}\right\}>L, \tag{24}$$

where

$$L=(1-c)^{\frac{1}{2}}\,{}_2F_1\left(-\frac{1}{2},1;\frac{\upsilon+p}{\sigma}+1;\frac{c}{c-1}\right).$$

The result is the best possible.

Proof Taking

$$\Theta(z)=\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}\quad(z\in\mathbb{U}), \tag{25}$$

then Θ is analytic in $\mathbb{U}$. After some calculations, we have

$$(1-\sigma)\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}F_{p,\upsilon}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}+\sigma\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q-1)}}{\delta(p,q-1)\,z^{p-q+1}}$$
$$=\Theta(z)+\left(\frac{\sigma}{p+\upsilon}\right)z\Theta'(z)\prec\sqrt{1+cz}.$$

By employing the same technique that was used in proving Theorem 1, the remaining part of the theorem can be proved. □

Theorem 3 *Let* $q\in\mathbb{N}_0$ *and* $p>q+\zeta$. *If* $f(z)\in\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta,\xi)$, *then* $f(z)\in\mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}(\zeta,\xi)$ *for* $|z|<R(p,q,\mu,\zeta,\xi)$ *where*

$$R(p,q,\mu,\zeta,\xi)=\min\{r>0:t(r)=0\}, \tag{26}$$

and

$$t(r)=1-\frac{2r}{(p-q-\zeta)\left|(1-\xi)(1-r)^2-\left|\xi+\frac{q+\zeta-\mu}{p-q-\zeta}\right|(1-r^2)\right|}.$$

Proof Assume that $f(z)\in\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta,\xi)$ and

$$u(z)=\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q)}}-\zeta\right), \tag{27}$$

then, $u(z)$ is analytic in $\mathbb{U}$ with $u(0) = 1$, $\Re\{u(z)\} > \xi$. After some computations, we have

$$u(z)+\frac{zu'(z)}{(p-q-\zeta)\,u(z)+(q+\zeta-\mu)}=\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q)}}-\zeta\right). \tag{28}$$

Letting $v(z)=\frac{u(z)-\xi}{1-\xi}$, then, $v(0) = 1$ with $\Re\{v(z)\} > 0$. Substituting in (28), we obtain

$$\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q)}}-\zeta\right)-\xi$$
$$=(1-\xi)\left[v(z)+\frac{zv'(z)}{(p-q-\zeta)\left[(1-\xi)\,v(z)+\xi\right]+(q+\zeta-\mu)}\right],$$

and so

$$\Re\left\{\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q)}}-\zeta\right)-\xi\right\}$$
$$\geq(1-\xi)\left[\Re\{v(z)\}-\frac{\left|zv'(z)\right|}{(p-q-\zeta)\left|(1-\xi)\,|v(z)|-\left|\xi+\frac{q+\zeta-\mu}{p-q-\zeta}\right|\right|}\right]$$
$$\geq(1-\xi)\left[\Re\{v(z)\}-\frac{\left|zv'(z)\right|}{(p-q-\zeta)\left|(1-\xi)\,\Re\{v(z)\}-\left|\xi+\frac{q+\zeta-\mu}{p-q-\zeta}\right|\right|}\right].$$

Applying the following well-known estimate [19]:

$$\Re\{v(z)\}\geq\frac{1-r}{1+r}\text{ and }\frac{\left|zv'(z)\right|}{\Re\{v(z)\}}\leq\frac{2nr^n}{1-r^{2n}}\ (|z|=r<1),$$

for $n = 1$, we get

$$\Re\left\{\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)^{(q)}}-\zeta\right)-\xi\right\}\geq(1-\xi)\,t(r)\,\Re\{v(z)\}.$$

It is easily seen that $t(r)$ is positive, if $|z| < R(p, q, \mu, \zeta, \xi)$, where R is given by (26). □

Theorem 4 *Let* $f(z) \in \mathcal{A}(p)$, $p > \mu$, $\gamma > 0$ *and*

$$\Re\left(\frac{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)\right)'}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}-\frac{\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p}f(z)}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\right)<\frac{\gamma}{p-\mu}, \tag{29}$$

then

$$\Re\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\right)^{-\frac{1}{2\gamma}}>\frac{1}{2}.$$

The result is sharp.

Proof From (8), (29) may be written as

$$\Re\left(1+\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)''}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}-\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\right)<\gamma,$$

or equivalently,

$$1+\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)''}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}-\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\prec-\frac{2\gamma z}{1-z}. \tag{30}$$

Letting

$$F(z)=\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\right)^{-\frac{1}{2\gamma}},$$

then, we can express (30) as

$$z\left(\log F(z)\right)'\prec z\left(\log\frac{1}{1-z}\right)'. \tag{31}$$

Fom [20], (31) implies to

$$F(z)\prec\frac{1}{1-z},$$

or equivalently,

$$\Re\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)}\right)^{-\frac{1}{2\gamma}}>\frac{1}{2}\ \ (z\in\mathbb{U}).$$

To show that the result is sharp, let

$$K(z)=z^p+\sum_{n=1}^{\infty}\frac{(p+1)_n(p+1-\mu+\eta)_n}{(p+1-\mu)_n(p+1-\lambda+\eta)_n}\frac{2\gamma\,(2\gamma-1)\ldots(2\gamma-n+1)}{n!}z^{p+n},$$

and so

$$\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}K(z)=z^p+\sum_{n=1}^{\infty}\frac{2\gamma\,(2\gamma-1)\ldots(2\gamma-n+1)}{n!}z^{p+n}$$
$$=z^p\,(1+z)^{2\gamma}.$$

It is easy to check that $K(z)$ satisfies (29) and

$$\Re\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}K(z)\right)'}{\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}K(z)}\right)^{-\frac{1}{2\gamma}}\to\frac{1}{2}$$

as $z\to 1^-$. This ends our proof. □

Theorem 5 *Consider that* $q\in\mathbb{N}_0$, $p>q+\zeta$ *and*

$$(p-q-\zeta)\,(1-A)+(q+\zeta-\mu)\,(1-B)\geq 0. \tag{32}$$

(i) Suppose that $\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p}f(z)\right)^{(q)}\neq 0$ *for all* $z\in\mathbb{U}^*:=\mathbb{U}\backslash\{0\}$, *then*

$$\mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}\,(\zeta;A,B)\subset\mathcal{S}_{p,q}^{\lambda,\mu,\eta}\,(\zeta;A,B)\,.$$

(ii) Also, assuming that

$$\frac{1-A}{1-B} \geq \frac{1}{p-q-\zeta} \max\left\{\frac{p-2q-2\zeta+\mu-1}{2}, -(q+\zeta-\mu)\right\}, \tag{33}$$

then

$$\mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}(\zeta;A,B) \subset \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta,\xi).$$

where the bound

$$\xi(A,B) = \frac{1}{p-q-\zeta}\left[\frac{p-\mu}{{}_2F_1\left(1,\frac{2(p-q-\zeta)(A-B)}{1-B}, p-\mu+1;\frac{1}{2}\right)} - (q+\zeta-\mu)\right], \tag{34}$$

is the best possible

Proof Let $f(z) \in \mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}(\zeta;A,B)$ and

$$G(z) = z\left(\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}}{\delta(p,q)\, z^{p-q}}\right)^{\frac{1}{p-q-\zeta}}. \tag{35}$$

Since $\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)} \neq 0$ for all $z \in \mathbb{U}^*$, then $G(z)$ is analytic in $\mathbb{U}$ with $G(0) = 0$ and $G'(0) = 1$. Differentiating both sides of (35) *logarithmically*, we get

$$\Psi(z) = \frac{zG'(z)}{G(z)} = \frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}} - \zeta\right). \tag{36}$$

Using (10) *in* (36), we have

$$(p-\mu)\frac{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)\right)^{(q)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}} = (p-q-\zeta)\Psi(z) + (q+\zeta-\mu). \tag{37}$$

Differentiating both sides of (37) *logarithmically*, we get

$$\frac{1}{p-q-\zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda+1,\mu+1,\eta+1,p} f(z)\right)^{(q)}} - \zeta\right) = \Psi(z) + \frac{z\Psi'(z)}{(p-q-\zeta)\Psi(z) + (q+\zeta-\mu)}.$$

Combining this identity together with $f(z) \in \mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}(\zeta;A,B)$, we obtain

$$\Psi(z) + \frac{z\Psi'(z)}{(p-q-\zeta)\Psi(z) + (q+\zeta-\mu)} \prec \frac{1+Az}{1+Bz} \equiv h(z).$$

We will use Lemma 2 for $\widetilde{\beta} = (p-q-\zeta)$, $\widetilde{\gamma} = (q+\zeta-\mu)$. Since $h(z)$ is a convex function in $\mathbb{U}$ and

$$\Re\left[(p-q-\zeta)\frac{1+Az}{1+Bz} + (q+\zeta-\mu)\right] > 0,$$

whenever (32) holds. Then $f(z) \in \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta;A,B)$ from Lemma 2. This completes the proof of (i). To prove (ii), we assume that (33) holds, then all the assumptions of Lemma 3 are satisfied for the above values of $\widetilde{\beta}$, $\widetilde{\gamma}$ and $\widetilde{\alpha} = \frac{1-A}{1-B}$. It follows that $\mathcal{S}_{p,q}^{\lambda+1,\mu+1,\eta+1}(\zeta;A,B) \subset \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta,\xi)$ where $\xi(A,B)$ given by (34) is the best possible. □

Theorem 6 *Assume that $q \in \mathbb{N}_0$, $p > q + \zeta$ and*

$$(p - q - \zeta)(1 - A) + (q + \zeta + \upsilon)(1 - B) \geq 0. \tag{38}$$

(i) Suppose that $\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q)} \neq 0$ for all $z \in \mathbb{U}^*$, then

$$\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B) \subset F_{p,\upsilon}\left(\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B)\right).$$

(ii) Also, assuming that

$$\frac{1 - A}{1 - B} \geq \frac{1}{p - q - \zeta} \max\left\{-\frac{q + 2\zeta + \upsilon + 1}{2}, -(p + \zeta + \upsilon)\right\}, \tag{39}$$

then

$$\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B) \subset \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; \tau(A, B)).$$

where the bound

$$\tau(A, B) = \frac{1}{p - q - \zeta}\left[\frac{2p - q + \upsilon}{{}_2F_1\left(1, \frac{2(p-q-\zeta)(A-B)}{1-B}, 2p - q + \upsilon; \frac{1}{2}\right)} - (p + \zeta + \upsilon)\right], \tag{40}$$

is the best possible.

Proof Let $f(z) \in \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B)$ and

$$H(z) = z\left(\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q)}}{\delta(p, q)\, z^{p-q}}\right)^{\frac{1}{p-q-\zeta}}. \tag{41}$$

Since $\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q)} \neq 0$ for all $z \in \mathbb{U}^*$, then $H(z)$ is analytic in $\mathbb{U}$ with $H(0) = 0$ and $H'(0) = 1$. Differentiating both sides of (41) *logarithmically*, we get

$$\Phi(z) = \frac{zH'(z)}{H(z)} = \frac{1}{p - q - \zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q)}} - \zeta\right). \tag{42}$$

Using (21) *in* (42), we have

$$(p + \upsilon)\frac{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} F_{p,\upsilon} f(z)\right)^{(q)}} = (p - q - \zeta)\Phi(z) + (q + \zeta + \upsilon). \tag{43}$$

Differentiating both sides of (43) *logarithmically*, we get

$$\frac{1}{p - q - \zeta}\left(\frac{z\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q+1)}}{\left(\mathcal{M}_{0,z}^{\lambda,\mu,\eta,p} f(z)\right)^{(q)}} - \zeta\right) = \Phi(z) + \frac{z\Phi'(z)}{(p - q - \zeta)\Phi(z) + (q + \zeta + \upsilon)}.$$

Combining this identity together with $f(z) \in \mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta; A, B)$, we obtain

$$\Phi(z) + \frac{z\Phi'(z)}{(p - q - \zeta)\Phi(z) + (q + \zeta + \upsilon)} \prec \frac{1 + Az}{1 + Bz} \equiv h(z).$$

We will use Lemma 2 for $\widetilde{\beta} = (p - q - \zeta)$, $\overline{\gamma} = (q + \zeta + \upsilon)$. Since $h(z)$ is a convex function in $\mathbb{U}$ and

$$\Re\left[(p - q - \zeta)\frac{1 + Az}{1 + Bz} + (q + \zeta + \upsilon)\right] > 0,$$

whenever (38) holds. Then $f(z) \in F_{p,\upsilon}\left(\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta;A,B)\right)$ from Lemma 2. This proves (i). To prove (ii), we assume that (39) holds, then all the assumptions of Lemma 3 are satisfied for $\widetilde{\beta}$, $\overline{\gamma}$ which stated above and $\widetilde{\alpha}=\frac{1-A}{1-B}$. It follows that

$$\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta;A,B)\subset\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta;\tau(A,B)),$$

where $\tau(A,B)$ given by (40) is the best possible □

Conclusion

In our present investigation, we have derived some subordination results of certain subclasses of multivalent analytic functions which are defined by a generalized fractional differintegral operator. We have also successfully considered inclusion relations for functions in the class $\mathcal{S}_{p,q}^{\lambda,\mu,\eta}(\zeta;A,B)$ and the images of these functions by the generalized Bernardi-Libera-Livingston integral operator.

Abbreviations

$\mathcal{A}(p)$: The class of analytic and multivalent functions in the open unit disc; $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$; $*$: Convolution of two power series; $\prec$: Subordination of two analytic functions in $\mathbb{U}$; ${}_2F_1(a,b;c;z)$ $(c\neq 0,-1,-2,\ldots)$: The well-known (Gaussian) hypergeometric function; $J_{0,z}^{\lambda,\mu,\eta,p}f(z)$: The generalized fractional derivative operator for $f\in\mathcal{A}(p)$; $F_{p,\upsilon}$: The generalized Bernardi-Libera-Livingston integeral operator

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Authors' contributions

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