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On a class of generalized φ-recurrent Sasakian manifold

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Abstract
The object of the present paper was to introduce the notion of hyper generalized φ-recurrent Sasakian manifold and quasi generalized φ-recurrent Sasakian manifold and study its various geometric properties. The existence of hyper generalized φ-recurrent Sasakian manifold and quasi generalized φ-recurrent Sasakian manifold is proved by giving a proper example.

Keywords: Hyper generalized recurrent, Quasi generalized recurrent, Generalized φ-recurrent, Generalized recurrent, Ricci-recurrent

AMS Subject Classification: Primary 53C25, 53C15, Secondary 53D15

Introduction

The notion of contact geometry has evolved from the mathematical formalism of classical mechanics [1]. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds [2]. An odd dimensional analog of Kaehler geometry is the Sasakian geometry. Sasakian manifolds were firstly studied by the famous geometer Sasaki [3] in 1960 and for long time focused on this. Sasakian manifolds have been extensively studied under several points of view in [4–8] and references therein.

The notion of local symmetry of a Riemannian manifold has been weakened by several authors in many ways to a different extent. As a mild version of local symmetry, Takahashi [9] introduced the notion of local φ-symmetry on a Sasakian manifold. Generalizing the idea of φ-symmetry, De et al. [10] introduced the concept of φ-recurrent Sasakian manifold. The notion of generalized recurrent manifolds was initiated by Dubey [11] and in [12] Shaikh et al. introduced the notion of generalized φ-recurrent Sasakian manifolds. Extending the notion of generalized φ-recurrent, Shaikh and Hui [13] introduced the concept of extended generalized φ-recurrent manifolds. In [14], Shashikala and Venkatesha studied generalized projective φ-recurrent Sasakian manifold. The extended generalized φ-recurrent property in Sasakian manifold was considered by Prakasha [15] and gave some important results.

A Riemannian manifold is called generalized recurrent if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes P, \quad (1)$$

where A and B are two non-vanishing 1-forms defined by $A(\cdot) = g(\cdot, \gamma_1)$, $B(\cdot) = g(\cdot, \gamma_2)$ and the tensor P is defined by

$$P(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{2}$$

for all $X, Y, Z \in TM$ and ∇ denotes the covariant differentiation with respect to the metric g. Here, γ_1 and γ_2 are vector fields associated with 1-forms A and B respectively. Especially, if the 1- form B vanishes, then (1) turns into the notion of recurrent manifold introduced by Walker [16]. A Riemannian manifold is called generalized φ-recurrent if its curvature tensor R satisfies the condition

$$\varphi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)P(X, Y)Z, \tag{3}$$

for all $X, Y, Z \in TM$, where P defined as in (2). Suppose the vector fields X, Y and Z are orthogonal to ξ, then the relation (3) reduces to the notion of locally generalized φ-recurrent manifolds.

A Riemannian manifold is called a generalized Ricci-recurrent manifold [17] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{4}$$

where A and B are non-vanishing 1-forms defined in (1). In particular, if $B = 0$, then (4) reduces to the notion of Ricci-recurrent manifold introduced by Patterson [18].

A Riemannian manifold is called a super generalized Ricci-recurrent manifold if its Ricci tensor S of type (0,2) satisfies the condition

$$\nabla S = \pi \otimes S + \rho \otimes g + \upsilon \otimes \eta \otimes \eta, \tag{5}$$

where π, ρ, and υ are non-vanishing unique 1-forms. In particular, if $\rho = \upsilon$, then (5) reduces to the notion of quasi-generalized Ricci-recurrent manifold introduced by Shaikh and Roy [19].

Recently, Shaikh and Patra [20] introduce a generalized class of recurrent manifolds called hyper generalized recurrent manifolds. In [19], Shaikh and Roy introduce a generalized class of recurrent manifolds called quasi generalized recurrent manifolds. The present paper deals with the study of both hyper generalized φ-recurrent and quasi generalized φ-recurrent property in Sasakian manifolds. The paper is organized as follows: The "Preliminaries" section is concerned with some preliminaries about Sasakian manifolds. In the "Hyper generalized φ-recurrent manifold," we introduce an extended form of hyper generalized recurrent manifolds called hyper generalized φ-recurrent manifolds. We study some geometric properties of this in Sasakian manifold and obtained some interesting results. We construct a proper example of a hyper generalized φ-recurrent Sasakian manifold which is neither φ-symmetric nor φ-recurrent in the "Example of hyper generalized φ-recurrent Sasakian manifold" section. In the "Quasi generalized φ-recurrent manifold" section, we introduce a generalized class of φ-recurrent manifold called quasi generalized φ-recurrent manifold and we study this property in Sasakian manifold and obtained some interesting results. Also, the existence of quasi generalized φ-recurrent Sasakian manifold is ensured by a proper example in the last section.

Preliminaries

In this section, we provide some general definition and basic formulas on contact metric manifolds and Sasakian manifolds which we will use in further sections. We may refer to [21–23] and references therein for more details and information about Sasakian geometry.

A (2n+1)-dimensional smooth connected manifold M is called almost contact manifold if it admits a triple (φ, ξ, η), where φ is tensor field of type $(1, 1)$, ξ is a global vector field and η is a 1-form, such that

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \qquad \varphi\xi = 0, \qquad \eta \circ \varphi = 0, \tag{6}$$

for all $X, Y \in TM$. If an almost contact manifold M admits a (φ, ξ, η, g), g being a Riemannian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{7}$$

then M is called an almost contact metric manifold. An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ with $d\eta(X, Y) = \Phi(X, Y)$, Φ being the fundamental 2-form of $M(\varphi, \xi, \eta, g)$ and is defined by $\Phi(X, Y) = g(X, \varphi Y)$, is a contact metric manifold and g is the associated metric. If, in addition ξ is a Killing vector field (equivalentely, $h = \frac{1}{2} L_\xi \varphi = 0$, where L denotes Lie differentiation), then the manifold is called K-contact manifold. It is well known that [2], if the contact metric structure (φ, ξ, η, g) is normal, that is, $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$ holds, then (φ, ξ, η, g) is Sasakian. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{8}$$

for all vector fields X and Y on M, where ∇ is Levi-Civita connection of g. A Sasakian manifold is always a K-contact manifold. The converse also holds when the dimension is three, but which may not be true in higher dimensions [24]. On any Sasakian manifold, the following relations are well known;

$$\nabla_X \xi = -\varphi X, \qquad (\nabla_X \eta)(Y) = g(X, \varphi Y), \tag{9}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{10}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{11}$$

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{12}$$

$$S(X, \xi) = 2n\eta(X) \qquad (or\ Q\xi = 2n\xi), \tag{13}$$

$$S(\varphi X, \varphi Y) = S(X, Y) - 2n\eta(X)\eta(Y), \tag{14}$$

for all $X, Y \in TM$, where R, S, and Q denotes the curvature tensor, Ricci tensor and Ricci operator respectively.

Definition 1 *A (2n+1)-dimensional Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y, where a and b are constants. If $b = 0$, then the manifold M is an Einstein manifold.

Hyper generalized φ-recurrent Sasakian manifold

Recently, the authors [20] studied hyper generalized recurrent manifolds and obtained several interesting results. By observing this work, we extend the notion called hyper generalized φ-recurrent manifolds. In this section, we study hyper generalized φ-recurrent Sasakian manifolds.

Definition 2 *A Sasakian manifold M is said to be a hyper generalized φ-recurrent Sasakian manifold if its curvature tensor R satisfies the condition*

$$\varphi^2\left((\nabla_W R)(X,Y)Z\right) = A(W)R(X,Y)Z + B(W)H(X,Y)Z, \tag{15}$$

for all $X,Y,Z \in TM$, where A and B are two non-vanishing 1-forms such that $A(X) = g(X,\rho_1)$, $B(X) = g(X,\rho_2)$ and the tensor H is defined by

$$H(X,Y)Z = S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY, \tag{16}$$

for all $X,Y,Z \in TM$. Here, ρ_1 and ρ_2 are vector fields associated with 1-forms A and B respectively. Especially, if the 1-form B vanishes, then (15) turns into the notion of φ-recurrent manifold.

Now we prove the following;

Theorem 1 *Let M be a hyper generalized φ-recurrent Sasakian manifold.*

(i) If the scalar curvature is zero everywhere on M, then M is Ricci recurrent.
(ii) If the scalar curvature is non-zero everywhere on M, then M is generalized Ricci recurrent.

Proof Let us consider hyper generalized φ-recurrent Sasakian manifold. In view of (6), Eq. (15) gives

$$\begin{aligned} &-(\nabla_W R)(X,Y)Z + \eta\left((\nabla_W R)(X,Y)Z\right)\xi \\ &\quad = A(W)R(X,Y)Z + B(W)H(X,Y)Z, \end{aligned} \tag{17}$$

this can be written as

$$\begin{aligned} &-g\left((\nabla_W R)(X,Y)Z,U\right) + \eta\left((\nabla_W R)(X,Y)Z\right)\eta(U) \\ &\quad = A(W)R(X,Y,Z,U) + B(W)g(H(X,Y)Z,U). \end{aligned} \tag{18}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be an orthonormal basis of the manifold. Plugging $X = U = e_i$ in (18) and taking summation over i, $1 \le i \le 2n+1$, and then using (16), we get

$$\begin{aligned} -(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta\left((\nabla_W R)(e_i,Y)Z\right)\eta(e_i) = (A(W) + (2n-1)B(W)) \\ S(Y,Z) + rB(W)g(Y,Z). \end{aligned} \tag{19}$$

The second term of left hand side in (19) reduces to

$$\sum_{i=1}^{2n+1} \eta\left((\nabla_W R)(e_i,Y)Z\right) = g\left((\nabla_W R)(\xi,Y)Z,\xi\right). \tag{20}$$

Using (9), (10) and the relation $g\left((\nabla_W R)(X,Y)Z,U\right) = -g\left((\nabla_W R)(X,Y)U,Z\right)$, we get

$$g\left((\nabla_W R)(\xi,Y)Z,\xi\right) = 0. \tag{21}$$

By virtue of (20) and (21), it follows from (19) that

$$(\nabla_W S)(Y,Z) = T(W)S(Y,Z) + \Psi(W)g(Y,Z), \tag{22}$$

where $T(W) = -(A(W) + (2n-1)B(W))$ and $\Psi(W) = -rB(W)$. In the above equation, we have hyper generalized φ-recurrent Sasakian manifold is Ricci recurrent (respectively generalized Ricci recurrent) if the scalar curvature is zero (respectively non-zero) everywhere on M. This completes the proof. □

Theorem 2 *A hyper generalized φ-recurrent Sasakian manifold M with non vanishing scalar curvature is an Einstein manifold and moreover the associated vector fields ρ_1 and ρ_2 of the 1-forms A and B respectively are co-directional.*

Proof Taking $Z = \xi$ in (22) and then using first term of (13), we obtain

$$(\nabla_W S)(Y,\xi) = -\{2nA(W) + (r + 2n(2n-1))B(W)\}\,\eta(Y). \tag{23}$$

At this point, we note that

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) + S(\nabla_W Y,\xi) + S(Y,\nabla_W \xi). \tag{24}$$

In view of (9) and first term of (13) in (24), it follows that

$$(\nabla_W S)(Y,\xi) = 2ng(Y,\varphi W) - S(Y,\varphi W). \tag{25}$$

Comparing (23) and (25), we get

$$2ng(Y,\varphi W) - S(Y,\varphi W) = -\{2nA(W) + (r + 2n(2n-1))B(W)\}\,\eta(Y). \tag{26}$$

Again taking φY instead of Y in (26) and using (6), (7) and (14), we have

$$S(Y,W) = 2ng(Y,W). \tag{27}$$

Substituting Y by ξ in (26), we get

$$A(W) = -\left[\frac{r + 2n(2n-1)}{2n}\right]B(W). \tag{28}$$

Contracting (27) over Y and W, we get

$$r = 2n(2n+1). \tag{29}$$

In view of (28) and (29), we have

$$A(W) = -4nB(W). \tag{30}$$

From (27) and (30), the theorem follows. □

It is well known that a Sasakian manifold is Ricci-semisymmetric if and only if it is an Einstein manifold. In fact, by Theorem 2, we have the following;

Corollary 2.1 *A hyper generalized φ-recurrent Sasakian manifold with non vanishing scalar curvature is Ricci-semisymmetric.*

Next, in a Sasakian manifold it can be easily verify that

$$(\nabla_W R)(X,Y)\xi = g(W,\varphi Y)X - g(W,\varphi X)Y + R(X,Y)\varphi W. \tag{31}$$

By virtue of (12), it follows from (31) that

$$\eta\left((\nabla_W R)(X,Y)\xi\right)=0. \tag{32}$$

It is well known that in a Sasakian manifold the following relation holds [8];

$$\begin{aligned}R(X,Y)\varphi Z &= g(\varphi X,Z)Y-g(Y,Z)\varphi X-g(\varphi Y,Z)X\\ &\quad+g(X,Z)\varphi Y+\varphi R(X,Y)Z,\end{aligned} \tag{33}$$

for any $X,Y,Z\in TM$. In view of (31) and (33), it follows that

$$(\nabla_W R)(X,Y)\xi=g(X,W)\varphi Y-g(Y,W)\varphi X+\varphi R(X,Y)W. \tag{34}$$

In view of (32) and (34), we obtain from (17) that

$$\begin{aligned}&-A(W)R(X,Y)\xi-B(W)\left\{2n\eta(Y)X-2n\eta(X)Y+\eta(Y)QX-\eta(X)QY\right\}\\ &=g(X,W)\varphi Y-g(Y,W)\varphi X+\varphi R(X,Y)W.\end{aligned} \tag{35}$$

In view of (10), (27) and (30), the above equation becomes

$$\varphi R(X,Y)W=g(Y,W)\varphi X-g(X,W)\varphi Y. \tag{36}$$

Operating φ on both sides of (36) and using (6), we get

$$R(X,Y)W=g(Y,W)X-g(X,W)Y. \tag{37}$$

Hence, we can state the following;

Theorem 3 *A hyper generalized φ-recurrent Sasakian manifold of non vanising scalar curvature is a space of constant curvature +1.*

Example of a hyper generalized φ-recurrent Sasakian manifold

In this section we give an example of a hyper generalized φ-recurrent Sasakian manifold. We consider three-dimensional manifold $M=\left\{(x,y,z)\in R^3,(x,y,z)\neq(0,0,0)\right\}$, where (x,y,z) are the standard coordinate in R^3. Let E_1,E_2,E_3 be three linearly independent vector fields in R^3 which satisfies

$$[E_1,E_2]=E_3,\quad [E_1,E_3]=-E_2,\quad [E_2,E_3]=2E_1.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1,$$
$$g(E_1,E_2)=g(E_1,E_3)=g(E_2,E_3)=0.$$

Let η be the 1-form defined by $\eta(W)=g(W,E_1)$ for any $W\in TM$. Let φ be the (1,1) tensor field defined by

$$\varphi E_1=0,\quad \varphi E_2=E_3,\quad \varphi E_3=-E_2.$$

Then using the linearity of η and g we have

$$\eta(E_1)=1,\qquad \varphi^2W=-W+\eta(W)E_1,$$
$$g(\varphi W,\varphi U)=g(W,U)-\eta(W)\eta(U),$$

for any $U,W\in TM$. Now for $E_1=\xi$, the structure (φ,ξ,η,g) defines an almost contact metric structure on M. Using the Koszula formula for the Riemannian metric g, we can straightforwardly calculate

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = 0,$$
$$\nabla_{E_2}E_1 = -E_3, \quad \nabla_{E_2}E_2 = 0, \quad \nabla_{E_2}E_3 = E_1,$$
$$\nabla_{E_3}E_1 = E_2, \quad \nabla_{E_3}E_2 = -E_1, \quad \nabla_{E_3}E_3 = 0.$$

From the above, it follows that the manifold under consideration is a Sasakian manifold of dimension 3. Using the above relations, we can straightforwardly calculate the non-vanishing components of the curvature tensor R as follows:

$$R(E_1,E_2)E_1 = -E_2, \quad R(E_1,E_2)E_2 = E_1, \quad R(E_1,E_3)E_1 = -E_3,$$
$$R(E_1,E_3)E_3 = E_1, \quad R(E_2,E_3)E_2 = E_3, \quad R(E_2,E_3)E_3 = -E_2$$

and the components which can be obtained from these by the symmetry properties. From the above, we can simply calculate the non-vanishing components of the Ricci tensor S and Ricci operator Q as follows:

$$S(E_1,E_1) = 2, \quad S(E_2,E_2) = S(E_3,E_3) = 0,$$
$$QE_1 = 2E_1, \quad QE_2 = QE_3 = 0.$$

Since $\{E_1,E_2,E_3\}$ forms a basis of the three-dimensional Sasakian manifold, any vector field $X, Y, Z \in TM$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3,$$
$$Y = a_2E_1 + b_2E_2 + c_2E_3,$$
$$Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers), $i = 1, 2, 3$. Now

$$R(X,Y)Z = [b_3(a_1b_2 - b_1a_2) + c_3(a_1c_2 - c_1a_2)]\,E_1 + [\,a_3(b_1a_2 - a_1b_2) + c_3(c_1b_2 - b_1c_2)]\,E_2 + [a_3(c_1a_2 - a_1c_2) + b_3(b_1c_2 - c_1b_2)]\,E_3, \tag{38}$$

and

$$H(X,Y)Z = [2(a_1b_2 - b_1a_2)b_3 + 2(a_1c_2 - c_1a_2)c_3]\,E_1 + [2(a_2b_1 - b_2a_1)a_3]\,E_2 + [2(a_2c_1 - c_2a_1)a_3]\,E_3. \tag{39}$$

In view of (39), we have the following;

$$(\nabla_{E_1}R)(X,Y)Z = 0, \tag{40}$$
$$(\nabla_{E_2}R)(X,Y)Z = [2(c_2b_1 - b_2c_1)b_3]\,E_1 + [\,2(a_1b_2 - b_1a_2)c_3 + 2(c_1b_2 - b_1c_2)a_3]\,E_2 + [2(a_2b_1 - b_2a_1)b_3]\,E_3, \tag{41}$$
$$(\nabla_{E_3}R)(X,Y)Z = [2(b_1c_2 - b_2c_1)c_3]\,E_1 + [2(a_1c_2 - c_1a_2)c_3]\,E_2 + [2(a_2c_1 - c_2a_1)b_3 + 2(c_1b_2 - b_1c_2)a_3]\,E_3. \tag{42}$$

From (41) to (42), we have

$$\varphi^2\left((\nabla_{E_i}R)(X,Y)Z\right) = p_iE_2 + q_iE_3 \quad \text{for } i = 1, 2, 3, \tag{43}$$

where

$$p_1 = 0, \quad q_1 = 0,$$
$$p_2 = 2\left[(b_1a_2 - a_1b_2)c_3 + (b_1c_2 - c_1b_2)a_3\right], \quad q_2 = 2(b_2a_1 - a_2b_1)b_3 = \frac{p_2(v_3u_1 - v_1u_3)}{v_2u_1 - v_1u_2},$$
$$p_3 = 2(c_1a_2 - a_1c_2)c_3, \quad q_3 = 2\left[(c_2a_1 - a_2c_1)b_3 + (b_1c_2 - c_1b_2)a_3\right] = \frac{p_3(v_3u_1 - v_1u_3)}{v_2u_1 - v_1u_2},$$

and

$$u_1 = \left[b_3(a_1b_2 - b_1a_2) + c_3(a_1c_2 - c_1a_2)\right], \quad u_2 = \left[a_3(b_1a_2 - a_1b_2) + c_3(c_1b_2 - b_1c_2)\right],$$
$$u_3 = \left[a_3(c_1a_2 - a_1c_2) + b_3(b_1c_2 - c_1b_2)\right], \quad v_1 = \left[2(a_1b_2 - b_1a_2)b_3 + 2(a_1c_2 - c_1a_2)c_3\right],$$
$$v_2 = \left[2(a_2b_1 - b_2a_1)a_3\right], \quad v_3 = \left[2(a_2c_1 - c_2a_1)a_3\right].$$

Let us now consider the components of the 1-forms as

$$\begin{aligned} A(E_1) &= 0, \quad B(E_1) = 0, \\ A(E_2) &= -\frac{p_2v_1}{v_2u_1 - v_1u_2}, \quad B(E_2) = \frac{p_2u_1}{v_2u_1 - v_1u_2}, \\ A(E_3) &= -\frac{p_3v_1}{v_2u_1 - v_1u_2}, \quad B(E_3) = \frac{p_3u_1}{v_2u_1 - v_1u_2}, \end{aligned} \tag{44}$$

where $v_2u_1 - v_1u_2 \neq 0$. From (15), we have

$$\varphi^2\left((\nabla_{E_i}R)(X,Y)Z\right) = A(E_i)R(X,Y)Z + B(E_i)H(X,Y)Z, \tag{45}$$

for $i = 1, 2, 3$. In view of (39), (40), (43) and (44), it can be easily shown that the manifold satisfies the relation (45). Hence the manifold under consideration is a hyper generalized φ-recurrent Sasakian manifold, which is not φ-recurrent. This leads to the following;

Theorem 4 *There exists a three-dimensional hyper generalized φ-recurrent Sasakian manifold, which is neither φ-symmetric nor φ-recurrent.*

Quasi generalized φ-recurrent Sasakian manifold

In the paper [19], the authors studied quasi generalized recurrent manifolds and obtain some interesting results. Motivated by this work, we extend the notion called quasi generalized φ-recurrent manifolds. In this section, we study quasi generalized φ-recurrent Sasakian manifolds.

Definition 3 *A Sasakian manifold M is said to be quasi generalized φ-recurrent manifold if its curvature tensor R satisfies the condition*

$$\varphi^2\left((\nabla_W R)(X,Y)Z\right) = C(W)R(X,Y)Z + D(W)F(X,Y)Z, \tag{46}$$

for all $X, Y, Z \in TM$, where C and D are two non-vanishing 1-forms such that $C(X) = g(X, \mu_1)$, $D(X) = g(X, \mu_2)$ and the tensor F is defined by

$$\begin{aligned} F(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \end{aligned} \tag{47}$$

for all $X, Y, Z \in TM$. Here μ_1 and μ_2 are vector fields associated with 1-forms C and D respectively. Especially, if the 1-form D vanishes, then (46) turns into the notion of φ-recurrent manifold.

Note: In view of (46) and (47), we say that locally quasi generalized φ-recurrent Sasakian manifold is a locally generalized φ-recurrent manifold.

We begin this section with the following:

Theorem 5 *A quasi generalized φ-recurrent Sasakian manifold M is an Einstein manifold and moreover the associated vector fields μ_1 and μ_2 of the 1-forms C and D respectively are co-directional.*

Proof Using the same steps as in the proof of Theorem 1, we get the relation

$$-(\nabla_W S)(Y,Z) = C(W)S(Y,Z) + D(W)(2n+1)g(Y,Z) + D(W)(2n-1)\eta(Y)\eta(Z). \tag{48}$$

Again using the same steps as in the Theorem 2, we get the equations

$$S(Y,W) = 2ng(Y,W), \qquad \text{and} \tag{49}$$

$$C(W) = \frac{1-4n^2}{2n}D(W), \tag{50}$$

for all Y, W. This completes the proof of the theorem. □

Equation (48) leads to the following:

Theorem 6 *A quasi generalized φ-recurrent Sasakian manifold is a super generalized Ricci-recurrent manifold.*

From (48), it follows that

$$-dr(W) = rC(W) + 2n(2n+3)D(W). \tag{51}$$

This leads to the following:

Theorem 7 *In a quasi generalized φ-recurrent Sasakian manifold, the 1-forms C and D are related by the Eq. (51).*

Corollary 7.1 *In a quasi generalized φ-recurrent Sasakian manifold with non-zero constant scalar curvature, the associated 1-forms C and D are related by*

$$rC(W) + 2n(2n+3)D(W) = 0.$$

Now suppose that quasi generalized φ-recurrent Sasakian manifold is quasi generalized Ricci-recurrent [19]. Then from (48) we have $2n+1 = 2n-1$, which is not possible. Therefore we can state the following;

Theorem 8 *A quasi generalized φ-recurrent Sasakian manifold can not be a quasi generalized Ricci-recurrent manifold.*

In view of (46) and (6), we obtain

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - C(W)R(X,Y)Z - D(W)F(X,Y)Z. \tag{52}$$

From (52) and second Bianchi identity we get

$$C(W)R(X,Y,Z,U)+D(W)F(X,Y,Z,U)+C(X)R(Y,W,Z,U)$$
$$+D(X)F(Y,W,Z,U)+C(Y)R(W,X,Z,U)+D(Y)F(W,X,Z,U)=0. \tag{53}$$

Contracting the above relation over Y and Z and using (47), we get

$$\begin{aligned}C(W)S(X,U)+D(W)&\left\{(2n+1)g(X,U)+(2n-1)\eta(X)\eta(U)\right\}\\&-C(X)S(W,U)-D(X)\left\{(2n+1)g(W,U)+(2n-1)\eta(W)\eta(U)\right\}\\&-C(R(W,X)U)+D(X)\left\{g(W,U)+\eta(W)\eta(U)\right\}-D(W)\{g(X,U)\\&+\eta(X)\eta(U)\}+D(\xi)\left\{\eta(X)g(W,U)-\eta(W)g(X,U)\right\}=0.\end{aligned} \tag{54}$$

Again contracting (54) over X and U and using (50) , we get

$$S(W,\mu_2)=\beta g(W,\mu_2)+\gamma\eta(W)\eta(\mu_2), \tag{55}$$

where $\beta=\frac{r}{2}+\frac{2n(2n^2-1)}{1-4n^2}$ and $\gamma=\frac{2n(1-4n)}{1-4n^2}$. Hence we can state the following;

Theorem 9 *In a quasi generalized φ-recurrent Sasakian manifold, the Ricci tensor S and vector field μ_2 are related by the Eq.* (55).

Definition 4 *[25] Let M be an almost contact metric manifold with Ricci tensor S. The $*$-Ricci tensor and $*$-scalar curvature of M are defined repectively by*

$$S^*(X,Y)=\sum_{i=1}^{2n+1}R(X,e_i,\varphi e_i,\varphi Y),\quad \textit{and}\quad r^*=\sum_{i=1}^{2n+1}S^*(e_i,e_i). \tag{56}$$

Definition 5 *[26] An almost contact metric manifold M is said to be weakly φ-Einstein if*

$$S^\varphi(X,Y)=\beta g^\varphi(X,Y),\quad X,Y\in TM,$$

for some function β. Here S^φ denotes the symmetric part of S^*, that is,

$$S^\varphi(X,Y)=\frac{1}{2}\left\{S^*(X,Y)+S^*(Y,X)\right\},\quad X,Y\in TM,$$

we call S^φ, the φ-Ricci tensor on M and the symmetric tensor g^φ is defined by $g^\varphi(X,Y)=g(\varphi X,\varphi Y)$. When β is constant, M is said to be φ-Einstein.

In a Sasakian manifold we know the following relation

$$(\nabla_W R)(X,Y)\xi=g(W,\varphi Y)X-g(W,\varphi X)Y+R(X,Y)\varphi W. \tag{57}$$

Using (57) and the relation $g\left((\nabla_W R)(X,Y)Z,\xi\right)=-g\left((\nabla_W R)(X,Y)\xi,Z\right)$ in (52), we have

$$\begin{aligned}(\nabla_W R)(X,Y)Z=&g(W,\varphi X)g(Y,Z)\xi-g(W,\varphi Y)g(X,Z)\xi-g(R(X,Y)\varphi W,Z)\xi\\&-C(W)R(X,Y)Z-D(W)F(X,Y)Z,\end{aligned} \tag{58}$$

from which it follows that

$$\begin{aligned}g((\nabla_W R)(X,Y)Z,U)=&g(W,\varphi X)g(Y,Z)\eta(U)-g(W,\varphi Y)g(X,Z)\eta(U)\\&+g(R(X,Y)Z,\varphi W)\eta(U)-C(W)g(R(X,Y)Z,U)\\&-D(W)g(F(X,Y)Z,U).\end{aligned} \tag{59}$$

Replacing Z by φZ in the foregoing equation, we obtain

$$g((\nabla_W R)(X,Y)\varphi Z,U) = g(W,\varphi X)g(Y,\varphi Z)\eta(U) - g(W,\varphi Y)g(X,\varphi Z)\eta(U) + g(R(X,Y)\varphi Z,\varphi W)\eta(U) - C(W)g(R(X,Y)\varphi Z,U) - D(W)g(F(X,Y)\varphi Z,U). \tag{60}$$

Since $g(R(X,Y)\varphi W,U) = g(R(X,Y)W,\varphi U)$ and $g((\nabla_W R)(X,Y)\varphi Z,U) = g((\nabla_W R)(X,Y)Z,\varphi U)$, using these equation in (60), we get

$$g((\nabla_W R)(X,Y)Z,\varphi U) = g(W,\varphi X)g(Y,\varphi Z)\eta(U) - g(W,\varphi Y)g(X,\varphi Z)\eta(U) + g(R(X,Y)\varphi Z,\varphi W)\eta(U) - C(W)g(R(X,Y)Z,\varphi U) - D(W)g(F(X,Y)\varphi Z,U). \tag{61}$$

Contracting (61) over Y and Z and using (47), we get

$$(\nabla_W S)(X,\varphi U) = -g(\varphi X,\varphi W)\eta(U) + S^*(X,W)\eta(U) - C(W)S(X,\varphi U) + D(W)g(X,\varphi U). \tag{62}$$

In view of (48), we have

$$S^*(X,W) = g(\varphi X,\varphi W) - \frac{(2n+2)D(W)}{\eta(U)}g(X,\varphi U). \tag{63}$$

Substituting $U = \xi$ in (63), we get

$$S^*(X,W) = g(\varphi X,\varphi W). \tag{64}$$

From (64) and Definition 5, we conclude that it is φ-Einstein. Hence we can state the following;

Theorem 10 *A quasi generalized φ-recurrent Sasakian manifold is an φ-Einstein manifold.*

In view of (7) and (64), we have the following;

Theorem 11 *A quasi generalized φ-recurrent Sasakian manifold is an $*$-η-Einstein manifold.*

Example of a quasi generalized φ-recurrent Sasakian manifold

In this section, we give an example of a quasi generalized φ-recurrent Sasakian manifold. We take the three-dimensional manifold $M = \{(x,y,z) \in R^3 : z > 0\}$, where (x,y,z) are the standard coordinates in R^3 . Let E_1, E_2, E_3 be linearly independent global frame on M given by

$$E_1 = \frac{\partial}{\partial y}, \quad E_2 = \frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1,E_1) = g(E_2,E_2) = g(E_3,E_3) = 1,$$
$$g(E_1,E_2) = g(E_1,E_3) = g(E_2,E_3) = 0.$$

Let η be the 1-form defined by $\eta(W) = g(W,E_3)$ for any $W \in TM$. Let φ be the (1,1) tensor field defined by

$$\varphi E_1 = -E_2, \quad \varphi E_2 = E_1, \quad \varphi E_3 = 0.$$

Then using the linearity of η and g we have

$$\eta(E_3) = 1, \qquad \varphi^2 W = -W + \eta(W)E_3,$$
$$g(\varphi W, \varphi U) = g(W, U) - \eta(W)\eta(U),$$

for any $U, W \in TM$. Then for $E_3 = \xi$, the structure (φ, ξ, η, g) defines an almost contact metric structure on M. Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[E_1, E_2] = -2E_3, \quad [E_1, E_3] = 0, \quad [E_2, E_3] = 0.$$

Using the Koszula formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_1} E_2 = -E_3, \quad \nabla_{E_1} E_3 = E_2,$$
$$\nabla_{E_2} E_1 = E_3, \quad \nabla_{E_2} E_2 = 0, \quad \nabla_{E_2} E_3 = -E_1,$$
$$\nabla_{E_3} E_1 = E_2, \quad \nabla_{E_3} E_2 = -E_1, \quad \nabla_{E_3} E_3 = 0.$$

From the above, it follows that the manifold under consideration is a Sasakian manifold of 3-dimension. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$R(E_1, E_2)E_1 = 3E_2, \quad R(E_1, E_3)E_1 = -E_3, \quad R(E_1, E_2)E_2 = -3E_1,$$
$$R(E_1, E_3)E_3 = E_1, \quad R(E_2, E_3)E_2 = -E_3, \quad R(E_2, E_3)E_3 = E_2$$

and the components which can be obtained from these by the symmetry properties. Since $\{E_1, E_2, E_3\}$ forms a basis of the three-dimensional Sasakian manifold, any vector field $X, Y, Z \in TM$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3,$$
$$Y = a_2E_1 + b_2E_2 + c_2E_3,$$
$$Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where $a_i, b_i, c_i \in R^+$ (the set of all positive real numbers), $i = 1, 2, 3$. Now

$$R(X, Y)Z = [(a_1c_2 - c_1a_2)c_3 + 3(b_1a_2 - a_1b_2)b_3]\, E_1 + [\,(b_1c_2 - c_1b_2)c_3 + 3(a_1b_2 - b_1a_2)a_3]\, E_2 - [(c_2a_1 - a_2c_1)a_3 + (c_2b_1 - b_2c_1)b_3]\, E_3, \tag{65}$$

and

$$F(X, Y)Z = [(a_1b_2 - a_2b_1)b_3 + 2(a_1c_2 - a_2c_1)c_3]\, E_1 + [\,(b_1a_2 - b_2a_1)a_3 + 2(b_1c_2 - b_2c_1)c_3]\, E_2 + 2\,[(c_1a_2 - c_2a_1)a_3 + (c_1b_2 - c_2b_1)b_3]\, E_3. \tag{66}$$

In view of (65), we have the following:

$$(\nabla_{E_1}R)(X, Y)Z = 4\,[(a_1b_2 - b_1a_2)(a_3E_3 - c_3E_1) + (a_1c_2 - c_1a_2)(a_3E_2 - b_3E_1)]\,, \tag{67}$$
$$(\nabla_{E_2}R)(X, Y)Z = 4\,[(a_1b_2 - b_1a_2)(b_3E_3 - c_3E_2) - (b_1c_2 - c_1b_2)(a_3E_2 + b_3E_1)]\,, \tag{68}$$
$$(\nabla_{E_3}R)(X, Y)Z = 0. \tag{69}$$

From (67)-(69), we have

$$\varphi^2((\nabla_{E_i}R)(X, Y)Z) = \alpha_iE_1 + \beta_iE_2 \quad \text{for } i = 1, 2, 3, \tag{70}$$

where

$$\alpha_1 = 4[c_3(a_1b_2-b_1a_2)+b_3(a_1c_2-c_1a_2)] = \frac{\beta_1(v_3u_1+u_3v_1)}{v_3u_2+u_3v_2}, \quad \beta_1 = -4a_3(a_1c_2-c_1a_2),$$

$$\alpha_2 = 4b_3(b_1c_2-c_1b_2) = \frac{\beta_2(v_3u_1+u_3v_1)}{u_3v_2+v_3u_2}, \quad \beta_2 = 4[c_3(a_1b_2-b_1a_2)+a_3(b_1c_2-c_1b_2)],$$

$$\alpha_3 = 0, \quad \beta_3 = 0,$$

and

$$u_1 = [(a_1c_2-c_1a_2)c_3+3(b_1a_2-a_1b_2)b_3], \quad u_2 = [(b_1c_2-c_1b_2)c_3+3(a_1b_2-b_1a_2)a_3],$$

$$u_3 = [(c_2a_1-a_2c_1)a_3+(c_2b_1-b_2c_1)b_3], \quad v_1 = [(a_1b_2-a_2b_1)b_3+2(a_1c_2-a_2c_1)c_3],$$

$$v_2 = [(b_1a_2-b_2a_1)a_3+2(b_1c_2-b_2c_1)c_3], \quad v_3 = 2[(c_1a_2-c_2a_1)a_3+(c_1b_2-c_2b_1)b_3].$$

Let us now consider the components of the 1-forms as

$$\begin{aligned} C(E_1) &= \frac{\beta_1 v_3}{v_3u_2+u_3v_2}, \quad D(E_1) = \frac{\beta_1 u_3}{v_3u_2+u_3v_2}, \\ C(E_2) &= \frac{\beta_2 v_3}{v_3u_2+u_3v_2}, \quad D(E_2) = \frac{\beta_2 u_3}{v_3u_2+u_3v_2}, \\ C(E_3) &= 0, \quad D(E_3) = 0, \end{aligned} \tag{71}$$

where $v_3u_2+u_3v_2 \neq 0$. From (46), we have

$$\varphi^2((\nabla_{E_i}R)(X,Y)Z) = C(E_i)R(X,Y)Z + D(E_i)F(X,Y)Z, \tag{72}$$

for $i = 1, 2, 3$. In view of (65), (66), (70), and (71), it can be easily shown that the manifold satisfies the relation (72). Hence, the manifold under consideration is a quasi generalized φ-recurrent Sasakian manifold, which is not φ-recurrent. This leads to the following:

Theorem 12 *There exists a three-dimensional quasi generalized φ-recurrent Sasakian manifold, which is neither φ-symmetric nor φ-recurrent.*

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References

1. Geiges, H. H.: A brief history of contact geometry and topology. Expo. Math. **19**(1), 25–53 (2001)
2. Blair, D. E.: Contact manifolds in Riemannian geometry. Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, Berlin-New York (1976)
3. Sasaki, S.: Lecture notes on almost contact manifolds-I. Mathematical Institute, Tohoku Univ (1965)
4. Chaki, M. C., Tarafdar, M.: On a type of Sasakian manifold. Soochow J. Math. **16**(1), 23–28 (1990)
5. De, U. C., Ozgur, C., Mondal, A. K.: On ϕ-quasiconformally symmetric Sasakian manifolds. Indag. Mathern. N.S. **20**(2), 191–200 (2009)
6. Ikawa, T., Kon, M.: Sasakian manifolds with vanishing contact Bochner curvature tensor and constant scalar curvature. Colloq. Math. **37**(1), 113–122 (1977)
7. Olszak, Z.: Certain property of the Ricci tensor on Sasakian manifolds. Collo. Math. **40**(2), 235–237 (1978)
8. Tanno, S.: Isometric immersions of Sasakian manifolds in spheres. Kōdai Math. Sem. Rep. **21**, 448–458 (1969)
9. Takahashi, T.: Sasakian φ-symmetric spaces. Tohoku Math. J. **29**, 91–113 (1977)
10. De, U. C., Shaikh, A. A., Biswas, S.: On φ-recurrent Sasakian manifolds. Novi Sad J. Math. **33**, 13–48 (2003)

11. Dubey, R. S. D.: Generalized recurrent spaces. Indian J. Pure Appl. Math. **10**(12), 1508–1513 (1979)
12. Shaikh, A. A., Ahmad, H.: On generalized φ-recurrent Sasakian manifolds. Appl. Math. **2**, 1317–1322 (2011)
13. Shaikh, A. A., Hui, S. K.: On extended generalized φ-recurrent-Kenmotsu manifolds. Publ. de l'Institut Math. **89**(103), 77–88 (2011)
14. Shashikala, S.: Venkatesha.: On generalized projective φ-recurrent Sasakian manifold. IOSR J. Math. **10**(1), 01–03 (2014)
15. Prakasha, D. G.: On extended generalized φ-recurrent Sasakian manifolds. J. Egypt. Math. Soc. **21**, 25–31 (2013)
16. Walker, A. A.: On Ruses spaces of recurrent curvature. Proc. Lond. Math. Soc. **52**, 36–64 (1950)
17. De, U. C., Guha, N., Kamilya, D.: On generalized Ricci-recurrent manifolds. Tensor N.S. **56**, 312–317 (1995)
18. Patterson, E. M.: Some theorems on Ricci-recurrent spaces. J. Lond. Math. Soc. **27**, 287–295 (1952)
19. Shaikh, A. A., Roy, I.: On quasi-generalized recurrent manifolds. Math. Pannonica. **21**(2), 251–263 (2010)
20. Shaikh, A. A., Patra, A.: On a generalized class of recurrent manifolds. Arch. Math. (BRNO). **46**, 71–78 (2010)
21. Arslan, K., De, U. C., Ozgur, C., Jun, J. B.: On class of Sasakian manifolds. Nihonkai Math. J. **19**, 21–27 (2008)
22. Boyer, C. P., Galicki, K.: On Sasakian-Einstein geometry. Int. J. Math. **11**, 873–909 (2000)
23. Kushwaha, A., Narain, D.: Some curvature properties on Sasakian manifolds. J. Int. Aca. Phy. Sci. **20**(4), 293–302 (2016)
24. Jun, J. B., Kim, U. K.: On 3-dimensional almost contact metric manifolds. Kyungpook Math. J. **34**(2), 293–301 (1994)
25. Blair, D. E., Yildirim, H.: On conformally flat almost contact metric manifolds. Mediterr. J. Math. **13**, 2759–2770 (2016)
26. Cho, J. T., Inoguchi, J. I.: On φ-Einstein contact Riemannian manifolds. Mediterr. J. Math. **7**, 143–167 (2010)

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