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The new study of some characterization of canal surfaces with Weingarten and linear Weingarten types according to Bishop frame

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Abstract

In this paper, we have a tendency to investigate a particular Weingarten and linear Weingarten varieties of canal surfaces according to Bishop frame in Euclidean 3-space E^3 satisfying some fascinating and necessary equations in terms of the Gaussian curvature, the mean curvature, and therefore the second Gaussian curvature. On the premise of those equations, some canal surfaces are introduced.

Keywords: Canal surfaces, Weingarten surfaces, Bishop frame, Gaussian curvature, Second Gaussian curvature

Mathematical Subject Classification (2010): 53A04, 53A05, 53B25, 53Z99

Introduction

A surface $\Upsilon = f(u, v)$ in Euclidean 3-space E^3 is said to be a (x, y)-Weingarten surface if for a pair (x, y), $x \neq y$ of the K Gaussian curvature, H mean curvature, and K_{II} second Gaussian curvature of a surface Υ satisfies $\psi(x, y) = 0$, where ψ is the Jacobi function defined by $\psi = xy - yx$.

Also Υ is said to be a (x, y)-linear Weingarten surface if for a pair (x, y), $x \neq y$ of the curvatures K, H, and K_{II} of a surface Υ satisfies $ax + by = c$, where $a, b, c \in R$ and $(a, b, c) \neq (0, 0, 0)$ (for more details see [1–7]). In 1863, Julius Weingarten was able to make a major step forward in the topic when he gave a class of surfaces isometric to a given surface of revolution. Surface for which there is a definite functional relation between the principal curvatures (which called curvature diagram) and also between the Gaussian and the mean curvatures is called Weingarten surface. The knowledge of the first fundamental form I and second fundamental form II of a surface facilitates the analysis and the classification of surface shape. Especially in recent years, the geometry of the second fundamental form II has become an important issue in terms of investigating intrinsic and extrinsic geometric properties of the surfaces. Very recent results concerning the curvature properties associated with II and other variational aspects can be found in [7–10].

One may associate to such a surface M geometrical objects measured by means of its second fundamental form, as second Gaussian curvature KII, respectively.

We are able to calculate KII of a surface by exchanging the components of the first fundamental form E, F, G by the components of the second fundamental form e, f,

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g severally in Brioschi formula that is given by Francesco Brioschi within the years of 1800s.

Identification of the curvatures associated with the second elementary variety of a surface opened a door to the analysis of the new categories of Weingarten surfaces. Since the center of the last century, many geometers have studied Weingarten surfaces and linear Weingarten surfaces and obtained several attention-grabbing and valuable results [11–13].

The (x, y)-Weingarten and (x, y)-linear Weingarten canal surfaces are a classical topic in differential geometry, as introduced by [14, 15].

The surface theory has been a preferred topic for several researchers in many aspects. Besides using curves and surfaces, canal surfaces are the foremost well-liked in pc-aided geometric style such as planning models of internal and external organs, getting ready of terrain infrastructures, constructing of mixing surfaces, reconstructing of shape, and robotic path designing [16, 17].

In this work, we study the (x, y)-Weingarten and (x, y)-linear Weingarten canal surfaces reference to Bishop frame in Euclidian 3-space that satisfy all the surfaces under consideration are assumed to be smooth, regular, and topologically connected unless generally expressed. In the "Geometric preliminaries" section, we clarify the basic conception of the Frenet frame and Bishop frame in the Euclidean 3-space E^3; also we give the parametric equation of the canal surface that will be used during this work. The "Canal surface according to Bishop frame in E^3" section is precise to prepare some fundamental facts about the first, second, and third fundamental forms, the Gaussian curvature, the mean curvature, and the second Gaussian curvature, in the "(x, y)-Weingarten canal surface according to Bishop frame in E^3" and "(x, y)-linear Weingarten canal surface according to Bishop frame in E^3" sections, the (x, y)-Weingarten and (x, y)-linear Weingarten canal surfaces are discussed.

Geometric preliminaries

Let $\lambda : I \to E^3$ be a unit speed curve parameterized by arc length u, denote $\{T(u), N(u), B(u)\}$ the moving Frenet frame along the curve $\lambda(u)$. Then, with the first and second curvatures, κ and τ respectively, the Frenet formulas are given by [18, 19]

$$\begin{pmatrix} T'(u) \\ N'(u) \\ B'(u) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(u) & 0 \\ -\kappa(u) & 0 & \tau(u) \\ 0 & \tau(u) & 0 \end{pmatrix} \begin{pmatrix} T(u) \\ N(u) \\ B(u) \end{pmatrix}, \tag{1}$$

where $\langle T(u), T(u)\rangle = \langle N(u), N(u)\rangle = \langle B(u), B(u)\rangle = 1$ and $\langle T(u), N(u)\rangle = \langle T(u), B(u)\rangle = \langle N(u), B(u)\rangle = 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative [20, 21].

Let us consider the Bishop frame $\{T(u), P(u), M(u)\}$ of the curve $\lambda(u)$ in the Euclidean 3-space E^3. In the residue, $T(u)$ is the unit tangent, $P(u)$ is the unit normal, and $M(u)$ is the unit binormal vector respectively. The Bishop frame $\{T(u), P(u), M(u)\}$ is expressed as [21, 22].

$$\begin{pmatrix} T'(u) \\ P'(u) \\ M'(u) \end{pmatrix} = \begin{pmatrix} 0 & k_1(u) & k_2(u) \\ -k_1(u) & 0 & 0 \\ -k_2(u) & 0 & 0 \end{pmatrix} \begin{pmatrix} T(u) \\ P(u) \\ M(u) \end{pmatrix}, \tag{2}$$

where $\langle T(u), T(u)\rangle = \langle P(u), P(u)\rangle = \langle M(u), M(u)\rangle = 1$ and $\langle T(u), P(u)\rangle = \langle T(u), M(u)\rangle = \langle P(u), M(u)\rangle = 0$. Here, we shall call $k_1(u)$ and $k_2(u)$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{pmatrix} T(u) \\ P(u) \\ M(u) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\nu(u) & -\sin\nu(u) \\ 0 & \sin\nu(u) & \cos\nu(u) \end{pmatrix} \begin{pmatrix} T(u) \\ N(u) \\ B(u) \end{pmatrix}, \tag{3}$$

where

$$\begin{aligned} \nu(u) &= arc\tan\left(\frac{k_2}{k_1}\right); k_1 \neq 0, \\ \tau(u) &= -\frac{d\nu(u)}{du}, \\ \kappa(u) &= \sqrt{k_1^2(u) + k_2^2(u)}, \end{aligned}$$

and

$$\begin{aligned} k_1(u) &= \kappa(u)\cos\nu(u), \\ k_2(u) &= \kappa(u)\sin\nu(u). \end{aligned}$$

The envelope of a 1-parameter family $u \rightarrow S^2(u)$ of spheres in E^3 is called a canal surface. The curve $\alpha(u)$ formed by the centers of the spheres is called the center or spine curve of the canal surface. The radius of the canal surface is the function r such that $r(u)$ is the radial of the sphere $S^2(u)$. Then, the canal surface Υ can be formed as follows:

$$f(u,v) = \alpha(u) + r(u)\left(-r'(u)\frac{r'(u)}{\|r'(u)\|^2} + \sqrt{\frac{1-(r'(u))^2}{\|r'(u)\|}}\Big(-\cos(v)N(u) + \sin(v)B(u)\Big)\right). \tag{4}$$

Suppose the center curve of the canal surface Υ is a unit speed curve $\alpha : (a,b) \rightarrow E^3$ with non-zero curvature. Then, the canal surface can be parametrized by the mapping

$$f(u,v) = \alpha(u) + r(u)\left(-r'(u)\,T(u) + \sqrt{1-(r'(u))^2}\Big(-\cos(v)N(u) + \sin(v)B(u)\Big)\right). \tag{5}$$

Canal surface according to Bishop frame in E^3

Consider a canal surface υ according to Bishop frame in E^3 taking the following form

$$f(u,v) = \alpha(u) + r(u)\left(-r'(u)\,T(u) + \sqrt{1-(r'(u))^2}\Big(-\cos(v)P(u) + \sin(v)M(u)\Big)\right). \tag{6}$$

where the center curve of a canal surface is a unit speed curve $\alpha : (a, b) \to E^3$. For subsequent use, we give some basic conclusions by direct calculations. By using Eqs. (2) and (6), and we may assume that $\alpha'(u) = T(u)$ and $r'(u) = -\cos\theta$ for some smooth function $\theta = \theta(u)$, then we have

$$f_u(u,v) = f_u^1(u,v)\,T(u) + f_u^2(u,v)\,P(u) + f_u^3(u,v)\,M(u), \tag{7}$$

where

$$\begin{aligned}
f_u^1(u,v) &= \sin\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big),\\
f_u^2(u,v) &= \cos\theta\big(\cos v\sin\theta + r(k_1 - \theta'\cos v)\big),\\
f_u^3(u,v) &= \cos\theta\big(\sin v\sin\theta - r(k_2 + \theta'\sin v)\big).
\end{aligned}$$

Then

$$f_v(u,v) = f_v^1(u,v)\,T(u) + f_v^2(u,v)\,P(u) + f_v^3(u,v)\,M(u), \tag{8}$$

where $f_v^1(u,v) = 0$, $f_v^2(u,v) = r\sin v\sin\theta$ and $f_v^3(u,v) = r\cos v\sin\theta$. Then, the quantities of the first fundamental form are given by

$$\begin{aligned}
E = \sin^2\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big)^2 + \cos^2\theta\big(\cos v\sin\theta\\
+ r(k_1 - \theta'\cos v)\big)^2 + \cos^2\theta\big(\sin v\sin\theta - r(k_2 + \theta'\sin v)\big)^2,\\
F = r^2\sin\theta\cos\theta(k_1\sin v + k_2\cos v),\\
G = r^2\sin^2\theta.
\end{aligned} \tag{9}$$

Equation (9) leads to $EG - F^2 = r^2\sin^2\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big)^2$. From Eqs. (2) and (8), the unit normal vector field to Υ according to Bishop frame in E^3 is given by

$$n = \frac{f_u(u,v)\times f_v(u,v)}{\|f_u(u,v)\times f_v(u,v)\|},$$

where

$$\begin{aligned}
f_u(u,v)\times f_v(u,v) = r\sin\theta\cos\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big)T(u)\\
- r\cos v\sin^2\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big)P(u)\\
+ r\sin v\sin^2\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big)M(u),
\end{aligned}$$

$$\|f_u(u,v)\times f_v(u,v)\| = r\sin\theta\big(\sin\theta + r(k_1\cos v - k_2\sin v - \theta')\big).$$

Then

$$n = \cos\theta\,T(u) - \cos v\sin\theta\,P(u) + \sin v\sin\theta\,M(u).$$

Moreover, we have

$$n_u = \sin\theta\big(k_1\cos\nu - k_2\sin\nu - \theta'\big)T(u) + \cos\theta\big(k_1 - \theta'\cos\nu\big)P(u) \\ + \cos\theta\big(k_2 + \theta'\sin\nu\big)M(u), \\ n_\nu = \sin\nu\sin\theta\, P(u) + \cos\nu\sin\theta\, M(u).$$

Then, the quantities of the second fundamental form are obtained by

$$\begin{aligned} L = &-\sin^2\theta\big(\sin\theta + r(k_1\cos\nu - k_2\sin\nu - \theta')\big)\big(k_1\cos\nu - k_2\sin\nu - \theta'\big) \\ &-\cos^2\theta\big(k_1 - \theta'\cos\nu\big)\big(\cos\nu\sin\theta + r(k_1 - \theta'\cos\nu)\big) \\ &-\cos^2\theta\big(k_2 + \theta'\sin\nu\big)\big(-\sin\nu\sin\theta + r(k_2 + \theta'\sin\nu)\big), \\ M = &-r\sin\theta\cos\theta\big(k_1\sin\nu + k_2\cos\nu\big), \\ N = &-r\sin^2\theta. \end{aligned} \tag{10}$$

The quantities of the third fundamental form are given by

$$\begin{aligned} e &= \sin^2\theta\big(-k_1\cos\nu + k_2\sin\nu + \theta'\big) + \cos^2\theta\big(k_1 - \theta'\cos\nu\big)^2 + \cos^2\theta\big(k_2 + \theta'\sin\nu\big)^2, \\ f &= \sin\theta\cos\theta\big(k_1\sin\nu + k_2\cos\nu\big), \\ g &= \sin^2\theta. \end{aligned} \tag{11}$$

From Eqs. (9), (10) and (11), we have the following lemma.

Lemma 1 *The first, the second, and the third fundamental forms of canal surfaces according to Bishop frame in E^3 satisfy*

$$L = \frac{E-p}{-r}, M = \frac{F}{-r}, N = \frac{G}{-r}, e = \frac{L-q}{-r}, f = \frac{M}{-r}, g = \frac{N}{-r}, \tag{12}$$

and

$$\begin{aligned} p &= \sin\theta\big(\sin\theta + r(k_1\cos\nu - k_2\sin\nu - \theta')\big) = \sin^2\theta - r\,q, \\ q &= \sin\theta\big(-k_1\cos\nu + k_2\sin\nu + \theta'\big). \end{aligned} \tag{13}$$

From Eq. (12), we see that $p \neq 0$ everywhere. From Lemma 1, the Gaussian curvature K and the mean curvature H of canal surface Υ according to Bishop frame in E^3 are given by

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-q}{r\,p}, \tag{14}$$

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{(2p - \sin^2\theta)}{2r^2\,p^2}. \tag{15}$$

The definition of the second Gaussian curvature is as follows: see [9].

$$K_{II} = \frac{1}{(LN - M^2)^2}\left\{\begin{vmatrix} -\frac{1}{2}L_{\nu\nu} + M_{u\nu} & \frac{1}{2}L_u & M_u - \frac{1}{2}L_\nu \\ M_\nu - \frac{1}{2}N_u & L & M \\ \frac{1}{2}N_\nu & M & N \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}L_\nu & \frac{1}{2}N_u \\ \frac{1}{2}L_\nu & L & M \\ \frac{1}{2}N_u & M & N \end{vmatrix}\right\}.$$

From Eq. (12), the second Gaussian curvature K_{II} of Υ can be written as

$$K_{II} = \frac{-1}{2048(r^2p^2q^2)} \sum_{i=0,j=-6}^{6} \sin\theta\Big(\xi_{i,j}\cos(i\nu + j\theta) + \eta_{i,j}\,\sin(i\nu + j\theta)\Big). \tag{16}$$

As we will see, it is not necessary to give the (long) expression of K_{II} but only the coefficients of higher order for the trigonometric functions.

Also, from Eq. (13), we have

$$r^2p^2q^2 = r^2\sin^4\theta\Big(\sin\theta + r\big(k_1\cos\nu - k_2\sin\nu - \theta'\big)\Big)^2. \tag{17}$$

From Eqs. (15), (16), and (17), we have the following theorem

Theorem 1 *The mean curvature H and the second Gaussian curvature K_{II} of non-developable canal surface $\Upsilon = f(u,v)$ according to Bishop frame in E^3 satisfy*

$$K_{II} = H + \frac{B}{2r^2p^2q^2}, \tag{18}$$

where

$$B = \frac{-1}{1024} \sum_{i=0,j=-6}^{6} \sin\theta\Big(\gamma_{i,j}\cos(i\nu + j\theta) + \delta_{i,j}\,\sin(i\nu + j\theta)\Big), \tag{19}$$

with coefficients $\gamma_{i,j}$ and $\delta_{i,j}$ are given.

As the same scheme of the above Lemma, we will write only the coefficients of higher order for the trigonometric functions in the expression of B.

Next, we can compute the partial derivatives of the Gaussian curvature K, the mean curvature H, and the second Gaussian curvature K_{II} of canal surface $\Upsilon = f(u,v)$ according to Bishop frame in E^3 for later use in the following section. From Eqs. (14) and (15), we have

$$\begin{aligned} K_u = {} & \tfrac{1}{r^2p^2}\sin^2\theta\Big(- 2r'k_1^2\cos^2\nu - 2rr'k_2^2\sin^2\nu + r'\theta'\sin\theta + r\theta'^2\cos\theta - 2rr'\theta'^2 \\ & + k_2\sin\nu\big(r\theta'\cos\theta + r'(\sin\theta - 4r\theta')\big) + k_1\cos\nu\big(- r'\theta'\cos\theta - r'\sin\theta \\ & + 4rr'(k_2\sin\nu + \theta')\big) + rk_1'\cos\nu\sin\theta - rk_2'\sin\nu\sin\theta - r\theta''\sin\theta\Big), \\ K_\nu = {} & \tfrac{-1}{rp^2}\sin^3\theta\Big(k_1\sin\nu + k_2\cos\nu\Big). \end{aligned} \tag{20}$$

$$\begin{aligned} H_u = {} & \tfrac{r'}{r^3p^3}\sin^2\theta\Big(\big(k_1\cos\nu - k_2\sin\nu - \theta'\big)\big(\sin\theta + 2r(k_1\cos\nu - k_2\sin\nu - \theta')\big)\Big) \\ & - \tfrac{1}{r^2p^3}\sin^3\theta\Big(\big(k_1\cos\nu - k_2\sin\nu - \theta'\big)\big(\sin\theta + 2r(k_1\cos\nu - k_2\sin\nu - \theta')\big)\Big) \\ & \times\sin^3\theta\Big(r'(k_1\cos\nu - k_2\sin\nu - \theta') + \theta'\cos\theta + r(k_1'\cos\nu - k_2'\sin\nu - \theta'')\Big) \\ & + \tfrac{1}{2r^2p^2}\sin^4\theta\big(k_1\cos\nu - k_2\sin\nu - \theta'\big)\Big(2r'(k_1\cos\nu - k_2\sin\nu - \theta') + \theta'\cos\theta \\ & + 2r(k_1'\cos\nu - k_2'\sin\nu - \theta'')\Big) + \tfrac{1}{2r^2p^2}\sin^3\theta\Big(\sin\theta\big(k_1\sin\nu + k_2\cos\nu\big)\big(\sin\theta \\ & + 2r(k_1\cos\nu - k_2\sin\nu - \theta')\big)\Big), \\ H_\nu = {} & \tfrac{1}{2r^2p^3}\sin^4\theta\Big(\big(k_1\sin\nu + k_2\cos\nu\big)\big(\sin\theta + 3r(k_1\cos\nu - k_2\sin\nu - \theta')\big)\Big). \end{aligned} \tag{21}$$

$$
\begin{aligned}
(K_{II})_u &= H_u + \tfrac{1}{2r^4p^4q^4}\left\{\left(r^2p^2q^2\right)\tfrac{\partial B}{\partial u} - B\tfrac{\partial\left(r^2p^2q^2\right)}{\partial u}\right\},\\
(K_{II})_v &= H_v + \tfrac{1}{2r^4p^4q^4}\left\{\left(r^2p^2q^2\right)\tfrac{\partial B}{\partial v} - B\tfrac{\partial\left(r^2p^2q^2\right)}{\partial v}\right\}.
\end{aligned} \tag{22}
$$

$$
\frac{\partial B}{\partial u} = \frac{-1}{2048}\sum_{i=0,j=-6}^{6}\left\{\begin{array}{c}\left((j\theta'\delta_{i,j}+\gamma_{i,j'})\sin\theta+\theta'\gamma_{i,j}\cos\theta\right)\cos\left(iv+j\theta\right)\\ \left((-j\theta'\gamma_{i,j}+\delta_{i,j'})\sin\theta+\theta'\delta_{i,j}\cos\theta\right)\sin\left(iv+j\theta\right)\end{array}\right\}
$$

$$
\frac{\partial B}{\partial v} = \frac{-1}{2048}\sum_{i=0,j=-6}^{6}\left(\left(i\delta_{i,j}\sin\theta\right)\cos\left(iv+j\theta\right)-\left(i\gamma_{i,j}\sin\theta\right)\sin\left(iv+j\theta\right)\right). \tag{23}
$$

$$
\begin{aligned}
\tfrac{\partial\left(r^2p^2q^2\right)}{\partial u} &= r\sin^3\theta\left(\sin\theta+r(k_1\cos v-k_2\sin v-\theta')\right)\left(k_1\cos v-k_2\sin v\right.\\
&\left.-\theta'\right)\left(-r'\theta'+r'\theta'\cos 2\theta-3r\theta'^2\sin 2\theta+4rr'\theta'^2\sin\theta+4r^2\theta'^3\cos\theta\right.\\
&+4rk_1^2\cos v(r'\sin\theta+r\theta'\cos\theta)+4rk_2^2\sin^2 v(r'\sin\theta+r\theta'\cos\theta)-r\theta''\\
&+r\theta''\cos 2\theta+4r^2\theta'\theta''\sin\theta+2k_1\cos v\left(r'\sin\theta(\sin\theta-4r(\theta'+k_2\sin\theta)\right)\\
&-r\left(-\theta'\cos\theta(3\sin\theta-4rk_2\sin v)+4r\theta'^2\cos\theta+2r\theta''\sin\theta\right)\left.\right)\\
&+k_2\sin v\big)(2r'\sin\theta(\sin\theta-4r\theta'+r(-3\theta'\sin 2\theta\\
&+8r\theta'^2\cos\theta+4r\theta''\sin\theta))\big),\\
\tfrac{\partial\left(r^2p^2q^2\right)}{\partial v} &= r^2\sin^4\theta\left(k_1\sin v+k_2\cos v\right)\left(-k_1\cos v+k_2\sin v+\theta'\right)\left(2\sin^2\theta\right.\\
&\left.+6r\sin\theta(k_1\cos v-k_2\sin v-\theta')+4r^2(-k_1\cos v+k_2\sin v+\theta')^2\right).
\end{aligned} \tag{24}
$$

(x, y)-Weingarten canal surface according to Bishop frame in E^3

In this section, we study the (x, y)-Weingarten canal surface.

Theorem 2 *A canal surface $\Upsilon = f(u, v)$ according to Bishop frame in E^3 is a (K, H)-Weingarten canal surface if and only if it is a tube or a surface of revolution.*

Proof A (K, H)-Weingarten canal surface Υ satisfies Jacobi equation

$$
H_u K_v = H_v K_u. \tag{25}
$$

□

From Eqs. (20) and (21), we have

$$
-\frac{r'\sin^2\theta\left(k_1\sin v k_2\cos v\right)\left(-k_1\cos v+k_2\sin v+\theta'\right)}{2r^4\left(\sin\theta+r(k_1\cos v-k_2\sin v-\theta')\right)^4},
$$

$$
\frac{r'\left(1-r'^2\right)}{2r^2p}K_vK. \tag{26}
$$

We assume that $K_v \neq 0$, $K \neq 0$. Then $r' = 0$; thus, the canal surface Υ is a tube. On the contrary, if Υ is a surface of revolution (i.e., $\kappa = 0 \Rightarrow k_1 = k_2 = 0$), then from Eqs. (13)–(15), we have

$$K=\frac{-r''}{r\left(1-r'^2-rr''\right)},\quad H=\frac{r''\left(1-r'^2-2rr''\right)}{2r^2\left(1-r'^2-rr''\right)^2}. \tag{27}$$

Thus, the partial derivatives of K and H are given by

$$K_u=\frac{-r'r''\left(-1+r'^2+4rr''\right)+rr'''\left(-1+r'^2\right)}{r^2\left(-1+r'^2+rr''\right)^2},$$
$$H_u=\frac{2r'r''\left((1-r'^2)^2+4rr''(-1+r'^2)+6r^2r''^2\right)-rr'''(-1+r'^2)(-1+r'^2+3rr'')}{2r^3\left(-1+r'^2+rr''\right)^3},$$
$$K_v=H_v=0. \tag{28}$$

From Eq. (28), the Jacobi Eq. (25) turns into an identity. On the other hand, if Υ is a tube, then from Eqs. (13)–(15), we have

$$K=\frac{-k_1\cos v+k_2\sin v}{r\left(1+r(k_1\cos v-k_2\sin v)\right)},\quad H=\frac{\left(k_1\cos v-k_2\sin v\right)\left(1+2r(k_1\cos v-k_2\sin v)\right)}{2r^2\left(1+r(k_1\cos v-k_2\sin v)\right)^2}. \tag{29}$$

$$K_u=\frac{k_1'\cos v-k_2'\sin v}{r\left(1+r(k_1\cos v-k_2\sin v)\right)^2},\quad K_v=-\frac{k_1\cos v+k_2\sin v}{r\left(1+r(k_1\cos v-k_2\sin v)\right)^2}. \tag{30}$$

$$H_u=-\frac{r\left(k_1'\cos v-k_2'\sin v\right)\left(1+3k_1\cos v-3k_2\sin v\right)}{2r^2\left(1+r(k_1\cos v-k_2\sin v)\right)^3},\quad H_v=\frac{r\left(k_1\sin v+k_2\cos v\right)\left(1+3k_1\cos v-3k_2\sin v\right)}{2r^2\left(1+r(k_1\cos v-k_2\sin v)\right)^3}. \tag{31}$$

By Eqs. (30) and (31), the Jacobi Eq. (25) is satisfied everywhere.

Theorem 3 *For a non-developable canal surface $\Upsilon=f(u,v)$ according to Bishop frame in E^3, the following statements are equivalent:*

i. Υ is locally a tube or a surface of revolution whose spine curve has a non-zero constant curvature
ii. Υ is a (H,K_{II})-Weingarten canal surface
iii. Υ is a (K,K_{II})-Weingarten canal surface

Proof Suppose that $\Upsilon=f(u,v)$ is a non-developable (H,K_{II})-Weingarten canal surface. Then, it satisfies

$$(K_{II})_u H_v=(K_{II})_v H_u. \tag{32}$$

□

By Eq. (22), we have

$$H_v\left\{\frac{\partial B}{\partial u}\left(r^2p^2q^2\right)-B\frac{\partial\left(r^2p^2q^2\right)}{\partial u}\right\}=H_u\left\{\frac{\partial B}{\partial v}\left(r^2p^2q^2\right)-B\frac{\partial\left(r^2p^2q^2\right)}{\partial v}\right\}. \quad (33)$$

Comparing the coefficients of the highest degree of Eq. (13) regarding $\cos^{13} v$ with the help of Eqs. (21)–(24), we have

$$6r^6r'k_1^{10}k_2^3\sin^{10}\theta=0.$$

Thus, we have $k_1=0$ or $k_2=0$ or both $k_1=0$ and $k_2=0$ or $r'=0$, $\sin\theta\neq 0$. If $r'=0$, then $r(u)$ =constant. From Eqs. (19), (22), (23), and (24) and comparing the coefficients of $\cos 6v$ in both sides, we have

$$\frac{3k_1^5k_2\left(k_1k_1'+k_2k_2'\right)}{\left(k_1\cos v-k_2\sin v\right)^2\left(1+rk_1\cos v-rk_2\sin v\right)^5}=0\,,$$

then we have

$$3k_1^5k_2\left(k_1k_1'+k_2k_2'\right)=0. \quad (34)$$

From Eq. (34), we have both $k_1=0$ and $k_2=0$ which implies that the Gaussian curvature $K=0$ (i.e., Υ is a (K,K_{II})-Weingarten canal surface.) Conversely, if the canal surface Υ is a tube or a surface of revolution whose spine curve has a non-zero constant curvature, then from Eqs. (17) and (18) we have

$$\left(r^2p^2q^2\right)=r^2r''^2\left(1-r'^2-rr''\right)^2,$$

$$B=\frac{1}{2}\Big(-2r''^2(-1+2rr''gfmj)\big(-r+(-1+r^2)r''\big)+r'^6(r''-rr''^2)+r'^2\big(r''+5rr''^2$$
$$+(2-6r^2)r''^3-rr''^2\big)+r'^4\big(-r''(2+3rr'')+2rr''^2\big)-rr'r^{(3)}+2rr'^3r^{(3)}-rr'^5r^{(3)}\Big),$$

$$K_{II}=H-\left(\frac{1}{4r^2r''^2\left(-1+r'^2+rr''\right)^2}\right)\begin{pmatrix}-r'^2r''\left(-1+r'^2\right)^2+r(-1+r'^2)\\+6r^2r''^3(-1+r'^2)+4r^3r''^4-2r''^3\\(-1+r'^2+2rr'')+rr'(-1+r'^2)^2\\(r'r''^2+r^{(3)})\end{pmatrix}, \quad (35)$$

$$\frac{\partial B}{\partial u}=\frac{1}{2}\Big(3r'^3r''^2(-1+r'^2)+8r'r''^4-r'^3r''^2(-1+r'^2)^2+6(-1+r'^2)r''^2r^{(3)}$$
$$-16r^3r''^3r^{(3)}-6r^2r''^2\big(4r'r''^2+3(-1+r'^2)r^{(3)}\big)+r\big(-r'^6(r''^2)'+r''r^{(3)}$$
$$(-5+16r''^2)-r''^2(r''^2)'-16r''r^{(3)}\big)+r'^4\big(2(r''^2)'-11r''r^{(3)}\big)$$
$$+r'(22r''^3-2r''^3-r^{(4)})+2r'^3(-12r''^3+4r''^3+r^{(4)})-r'^5(6r''^3+r^{(4)})\Big),$$

$$\frac{\partial B}{\partial v}=0,\ \frac{\partial\left(r^2p^2q^2\right)}{\partial u}=2rr''(-1+r'^2+rr'')\big(r'r''(-1+r'^2+4rr'')$$
$$+r(-1+r'^2+2rr'')r'''\big),\ \frac{\partial\left(r^2p^2q^2\right)}{\partial v}=0,\ (K_{II})_v=0,$$

$$
(K_{II})_u = \left(\frac{-1}{4r^2 r''^3\big(-1+r'^2+rr''\big)^2}\right)
\left(\begin{array}{c}
r'^9 r''^2(-2+rr'') + 3r^2 r'^4 r''^2(1-3rr'')r^{(3)} + r^2 r'^6 r''^2(-1+2rr'')r^{(3)} \\
r^2 r''^2(-1+rr'')(-1+2rr'')r^{(3)} + r^2 r'^2 r''^2(-3-2rr''(-5+rr''))r^{(3)} \\
r'^7\big(6r''^2 - 2rr''^3 + r^2 r''^4 + 2r^2 r^{(3)^2} - r^2 r'' r^{(4)}\big) + rr' \\
4r''^3(-1+rr'')(1+rr''(-1+2rr'')) + 2r(-1+2rr'')r^{(3)^2} + rr''(1-rr'')r^{(4)} \\
r'^3\big(2r''^2(1+4rr''(-1+rr'')(-1+2rr'')) + 2r^2(3-4rr'')r^{(3)^2} + r^2 r''\big) \\
(-3+2rr'')r^{(4)} - r'^5\big(3rr''^3 - 15r^2 r''^4 + 6r^3 r''^5 + 6r^2 r^{(3)^2}\big) \\
-r^2 r''\big(4rr^{(3)^2} + 3r^{(4)} + r''^2(6+r^3 r^{(4)})\big)
\end{array}\right).
$$

By using Eq. (28) and the above equations, the Jacobi Eq. (25) converts to an identity. In case that the canal surface Υ is a tube whose spine curve has a non-zero constant curvature, it satisfies the Jacobi Eq. (25). Using a similar substantiation advanced above, we have the same results for the cases of (K, K_{II})-Weingarten canal surfaces and (K, K_{II}) Weingarten canal surfaces according to Bishop frame in E^3. This completes the proof.

(x, y)-linear Weingarten canal surface according to Bishop frame in E^3

First, we study some special (x, y)-linear Weingarten canal surface with Bishop frame in E^3 including developable canal surfaces, minimal canal surfaces, and the canal surfaces with vanishing second Gaussian curvature.

Remark 1 *The (x, y)-linear Weingarten canal surfaces are considered as a natural generalization of canal surfaces with constant Gaussian curvature, constant mean curvature, or constant second Gaussian curvature.*

Theorem 4 *A canal surface $\Upsilon = f(u, v)$ according to Bishop frame in E^3 is developable if and only if it is a circular cylinder or a circular cone.*

Proof $\Upsilon = f(u, v)$ is developable if and only if its Gaussian curvature $K = 0$. From Eq. (14), we have $q = 0$. Also, from Eq. (13), we get

$$
q = \sin\theta(-k_1\cos v + k_2\sin v + \theta') = \sin\theta(-k_1\cos v + k_2\sin v) + r'' = 0. \tag{36}
$$

□

From Eq. (36), we have $r'' = 0$ and $k_1 = k_2 = 0$ (i.e., $\kappa = 0$). Therefore, $r(u) = c_1 u + c_2$, where c_1 and c_2 are constants and $c_1 \neq \pm 1$ or (else $\sin\theta$, a contradiction). Then, Υ is a circular cylinder if $c_1 = 0$ or a circular cone if $c_1 \neq 0$ and $c_1 \neq \pm 1$.

Theorem 5 *A canal surface $\Upsilon = f(u, v)$ according to Bishop frame in E^3 is minimal if and only if it is a catenoid.*

Proof Since Υ is minimal if and only if its mean curvature $H = 0$, then Eq. (15) implies

$$
\begin{aligned}
&2p - \sin^2\theta = 0, \\
&\sin^2\theta + 2r\sin\theta(k_1\cos v - k_2\sin v) - 2rr'' = 0.
\end{aligned} \tag{37}
$$

□

Then, we have $\sin^{\theta} -2rr'' = 0$ and $2r\sin\theta(k_1\cos\nu - k_2\sin\nu)$. Since $r \neq 0$ and $\sin\theta$, then $k_1\cos\nu - k_2\sin\nu = 0$ implies that $\kappa = 0$ and Υ is a surface of revolution. It is well known that the only minimal surface of revolution is the catenoid.

It is recognized that a minimal surface satisfied that $K_{II} = 0$. However, a surface with vanishing second Gaussian curvature is not necessary to be minimal [6].

Theorem 6 *A non-developable canal surface $\Upsilon = f(u,\nu)$ according to Bishop frame in E^3 with vanishing second Gaussian curvature $K_{II} = 0$ is a surface of revolution which satisfies*

$$\begin{aligned}&-2rr''^2(-1+rr'')(-1+2rr'')+r'^6(r''-rr''^2)+r'^2(r''+5rr''^2-6r^2r''^3-rr''^2)\\&\quad+r'^4(-r''(2+3rr'')+2rr''^2)+(-rr'+2rr'^3-rr'^5)r'''=0.\end{aligned} \tag{38}$$

Proof When $K_{II} = 0$, we have from Eq. (19)

$$H = -\tfrac{B}{2r^2p^2q^2}, B = -q^3(2p - \sin^2\theta),$$
$$B = \tfrac{-1}{2048}\sum_{i=0,j=-6}^{6} \sin\theta\Big(\gamma_{i,j}\ \cos\left(i\nu + j\theta\right) + \delta_{i,j}\ \sin\left(i\nu + j\theta\right)\Big).$$

□

All the coefficients of Eq. (19), $\gamma_{i,j}$ and $\delta_{i,j}$, are vanished exept $\gamma_{1,0}$ and $\delta_{1,0}$, then we have $k_1 = k_2 = 0$ which implies that $\kappa = 0$. Then, the canal surface is a surface of revolution. Furthermore, by Eqs. (27) and (35), we have

$$\begin{aligned}&\tfrac{1}{2}\Big(-2rr''^2(-1+rr'')(-1+2rr'')+r'^6(r''-rr''^2)+r'^2(r''+5rr''^2-6r^2r''^3-rr''^2)\\&\quad+r'^4(-r''(2+3rr'')+2rr''^2)+(-rr'+2rr'^3-rr'^5)r'''\Big)=0.\end{aligned}$$

Now, we study some properties of $(x;y)$-linear Weingarten canal surfaces. Without losing generality, we may assume that $c = 1$ in $ax + by = c$.

Theorem 7 *A canal surface $\Upsilon = f(u,\nu)$ according to Bishop frame in E^3 is a (K,H)-linear Weingarten canal surface if and only if it is one of the following surfaces:*

i. *Υ tube with radius $r = \frac{b}{a}$,*
i.. *Υ surface of revolution such as*

$$f(u,\nu) = \Big(-r(u)\cos\nu\sin\theta, \sin\nu\sin\theta, u + \cos\theta\Big),$$

where $r(u)$ is given by Eq. (41).

Proof A (K,H)-linear Weingarten canal surface satisfied

$$aK + bH = 1,$$

where $a, b \in R$, and $(a, b) \neq (0, 0)$. From Eqs. (14) and (15), we have

$$\begin{array}{c} (-b+ar)\big(\sin\theta(k_1\cos\nu-k_2\sin\nu)\big) \\ \left(\begin{array}{c} 2r\big(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''\big)^2+\sin\theta(k_1\cos\nu-k_2\sin\nu) \\ \big(-2r^2\big(1-r'^2+r(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''^2)\big)^2+(-b+2ar) \\ \left(\begin{array}{c}(1-r'^2)(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''^2)-(-b+2ar) \\ (1-r'^2)r''+2r^2(1-r'^2+rr'')^2\big)\end{array}\right) \end{array}\right) \\ -(-b-2ar)(-1+r'^2)r''+2r(-b+ar)r''^2-2r^2(-1+r'^2+rr'')^2 \end{array} = 0.$$

Therefore, we get

$$\begin{array}{c} (-b+ar)\big(\sin\theta(k_1\cos\nu-k_2\sin\nu)\big) \\ \left(\begin{array}{c} 2r\big(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''\big)^2+\sin\theta(k_1\cos\nu-k_2\sin\nu) \\ \big(-2r^2\big(1-r'^2+r(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''^2)\big)^2+(-b+2ar) \\ \left(\begin{array}{c}(1-r'^2)(\sin\theta(k_1\cos\nu-k_2\sin\nu)-r''^2)-(-b+2ar) \\ (1-r'^2)r''+2r^2(1-r'^2+rr'')^2\big)\end{array}\right) \end{array}\right) \end{array} = 0, \quad (39)$$

and

$$-(-b-2ar)(-1+r'^2)r''+2r(-b+ar)r''^2-2r^2(-1+r'^2+rr'')^2=0. \quad (40)$$

□

Case 1 *From Eq. (39), if $k_1 = k_2 = 0$, this mean that $\kappa = 0$. Thus, Υ is a surface of revolution and its radial function satisfies Eq. (40)*

$$-(b-2ar)(-1+r'^2)r''+2r(-b+ar)r''^2-2r^2(-1+r'^2+rr'')^2=0.$$

Solving the above equation, we get

$$u = c_2 \pm \int \sqrt{\frac{c_1 + r(b-ar+r^3)}{r(b-ar+r^3)}}dr, \quad (41)$$

where c_1 and c_2 are constants [23]. Since $\kappa = 0$, we assume that the spine curve can be taken as the following form $\alpha(u) = (0, 0, u)$ and $T(u) = (0, 0, 1)$, $P(u) = (1, 0, 0)$, and $M(u) = (0, 1, 0)$. Then, $f(u, v)$ can be expressed by

$$f(u,v) = \Big(-r(u)\cos\nu\sin\theta, \sin\nu\sin\theta, u+\cos\theta\Big),$$

where $r(u)$ is given by Eq. (41).

Case 2 *If $\kappa \neq 0$, then $-b+ar = 0$. Hence, $r = \frac{b}{a}$ is a non-zero constant, $f(u, v)$ is a tube. Note that $\Upsilon = f(u, v)$ is a circular cylinder if $k_1 = k_2 = \kappa = 0$ and $-b+ar = 0$.*

Corollary 1 *The canal surface $\Upsilon = f(u, v)$ according to Bishop frame in E^3 which has non-zero constant Gaussian curvature is a surface of revolution such as*

$$f(u,v) = \Big(-r(u)\cos\nu\sin\theta, \sin\nu\sin\theta, u+\cos\theta\Big),$$

where $r(u)$ is given by Eq. (42).

Proof By Remark 1 and Theorem 7 with $b = 0$, Υ has non-zero constant Gaussian curvature $K = \frac{1}{a}$. Then, Υ cannot be a tube and it is a surface of revolution. Additionally, by a similar development as was given in Theorem 7, Υ can be expressed by

$$f(u,v) = \Big(-r(u)\cos v \sin\theta, \sin v \sin\theta, u + \cos\theta \Big),$$

in which $r(u)$ is given by

$$u = c_2 \pm \int \sqrt{\frac{c_1 + r(-ar + r^3)}{r(-ar + r^3)}} dr, \tag{42}$$

where $c_1 and c_2$ are constants [23]. □

Corollary 2 *The canal surface $\Upsilon = f(u,v)$ according to Bishop frame in E^3 which has non-zero constant mean curvature is a surface of revolution such as*

$$f(u,v) = \Big(-r(u)\cos v \sin\theta, \sin v \sin\theta, u + \cos\theta \Big),$$

where $r(u)$ is given by Eq. (43).

Proof By Remark 1 and Theorem 7 with $a = 0$, $f(u,v)$ has non-zero constant mean curvature $H = \frac{1}{b}$. Similarly, as Corollary 1, $f(u,v)$ is a surface of revolution and it can be expressed by

$$f(u,v) = \Big(-r(u)\cos v \sin\theta, \sin v \sin\theta, u + \cos\theta \Big),$$

in which, $r(u)$ is given by

$$u = c_2 \pm \int \sqrt{\frac{c_1 + r(b + r^3)}{r(b + r^3)}} dr, \tag{43}$$

where $c_1 and c_2$ are constants [23]. □

Theorem 8 *A non-developable canal surface $\Upsilon = f(u,v)$ according to Bishop frame in E^3 is a (H, K_{II})-linear Weingarten canal surface if and only if it is an open part of a surface of revolution satisfies*

$$\begin{aligned} r''\big(-ar'^2(-1 + 2r'^2 + r'^4) + r(-2(a+2r) + 4(a+2r)r'^2 - (a+4r)r'^4 \\ + ar'^6)r'' - 2(b + r^2(3a+4r))(-1+r'^2)r''^2 - 4r(b + r^2(a+r))r''^3\big) \\ -arr'(-1+r'^2)^2 r^{(3)} = 0, \end{aligned} \tag{44}$$

where $a, b \in R$, and $(a,b) \neq (0,0)$.

Proof Suppose $f(u,v)$ is a (H, K_{II})-linear Weingarten canal surface. It satisfies

$$aK_{II} + bH = 1,$$

where $a, b \in R$, and $(a,b) \neq (0,0)$. From Eqs. (15) and (16), we have

$$\begin{aligned} \frac{-a}{2048(r^2p^2q^2)} \sum_{i=0,j=-6}^{6} \sin\theta \Big(\xi_{i,j} \cos(iv + j\theta) + \eta_{i,j} \sin(iv + j\theta) \Big) \\ + b\frac{q(2p - \sin^2\theta)}{2r^2p^2} = 1. \end{aligned}$$

Then

$$-a\sum_{i=0,j=-6}^{6}\sin\theta\ \left(\xi_{i,j}\cos\left(i\nu+j\theta\right)+\eta_{i,j}\sin\left(i\nu+j\theta\right)\right)$$
$$=2048\left(r^2p^2q^2\right)\left(1-b\frac{q(2p-\sin^2\theta)}{2r^2p^2}\right),$$

$$\begin{aligned}&-a\textstyle\sum_{i=0,j=-6}^{6}\sin\theta\Big(\xi_{i,j}\ \cos\left(i\nu+j\theta\right)+\eta_{i,j}\sin\left(i\nu+j\theta\right)\Big)\\
&=-512\Big(\sin^4\theta\left(k_1\cos\nu-k_2\sin\nu-\theta'\right)^2\left(-2r^2+2r^2\cos2\theta-2brk_1^2-2r^4k_1^2\right.\\
&+\sin\left(\nu+\theta\right)(-bk_1-4r^3k_1)+\sin\left(\nu-\theta\right)\left(bk_1+4r^3k_1\right)-2brk_2^2-2r^4k_2^2\\
&+\cos\left(\nu+\theta\right)\left(-bk_2-4r^3k_2\right)+\cos\left(\nu-\theta\right)\left(bk_2+4r^3k_2\right)+\sin2\nu(4brk_1k_2\\
&+4r^4k_1k_2)+\cos2\nu(-2brk_1^2-2r^4k_1^2+2brk_2^2+2r^4k_2^2)-4br\theta'^2-4r^4\theta'^2\\
&\left.+\sin\theta(2b\theta'+8r^3\theta')+\cos\nu\left(8brk_1\theta'+8r^4\theta'k_1\right)\sin\nu(-8br\theta'k_2-8r^4\theta'k_2)\right)\Big).\end{aligned}\tag{45}$$

By comparing the coefficient of $\cos5\nu$, $\sin5\nu$ in Eq. (45), we have

$$\begin{gathered}a\sin\theta\,\xi_5=0,\\
a\sin\theta\big(-92r^2k_1^5+920r^2k_1^3k_2^2-460r^2k_1k_2^4\big)=0,\\
a\sin\theta\,\eta_5=0,\\
a\sin\theta\big(460r^2k_1^4k_2-920r^2r^2k_1^2k_2^3+92k_2^5\big)=0.\end{gathered}$$

It is follows that $k_1=k_2=0$, this means $\kappa=0$. Thus, the surface is a surface of revolution. From Eqs. (27) and (37), we have

$$\begin{aligned}&r''\big(-ar'^2(-1+2r'^2+r'^4)+r(-2(a+2r)+4(a+2r)r'^2-(a+4r)r'^4+ar'^6)r''\\
&-2(b+r^2(3a+4r))(-1+r'^2)r''^2-4r(b+r^2(a+r))r''^3\big)-arr'(-1+r'^2)^2r^{(3)}=0,\end{aligned}$$

□

Theorem 9 *A non-developable canal surface $\Upsilon=f(u,v)$ according to Bishop frame in E^3 is a (K,K_{II})-linear Weingarten canal surface if and only if it is an open part of a surface of revolution satisfies*

$$\begin{aligned}&2r''\big(a(r'-r'^3)^2-r(-1+r'^2)(-2(a+2r)+2(a+2r)r'^2+ar'^4)r''\\
&-2r(-2b+3ar+4r^2)(-1+r'^2)r''^2-4r^2(-b+r(a+r))r''^3\big)\\
&-2arr'(-1+r'^2)^2r^{(3)}=0,\end{aligned}\tag{46}$$

where $a,b\in R$, and $(a,b)\neq(0,0)$.

Proof Suppose $f(u,v)$ is a (K,K_{II})-linear Weingarten canal surface. It satisfies

$$aK_{II}+bK=1,$$

where $a,b\in R$, and $(a,b)\neq(0,0)$. From Eqs. (15) and (16), we have

$$\begin{aligned}&\tfrac{-a}{2048(r^2p^2q^2)}\textstyle\sum_{i=0,j=-6}^{6}\sin\theta\Big(\xi_{i,j}\ \cos\left(i\nu+j\theta\right)+\eta_{i,j}\ \sin\left(i\nu+j\theta\right)\Big)+b\left(\tfrac{-q}{rp}\right)=1\\
&-a\textstyle\sum_{i=0,j=-6}^{6}\sin\theta\Big(\xi_{i,j}\ \cos\left(i\nu+j\theta\right)+\eta_{i,j}\ \sin\left(i\nu+j\theta\right)\Big)\\
&=2048(r^2p^2q^2)\left(1-b\left(\tfrac{-q}{rp}\right)\right),\end{aligned}$$

$$
\begin{aligned}
&- a \sum_{i=0,j=-6}^{6} \sin\theta\Big(\xi_{i,j}\, \cos\left(i\nu + j\theta\right) + \eta_{i,j}\, \sin\left(i\nu + j\theta\right)\Big)\\
&= 2048\Big(r\sin^4\theta(k_1\cos\nu - k_2\sin\nu - \theta')^2\big(\sin\theta + rk_1\cos\nu - rk_2\sin\nu\\
&- r\theta'\big)\big(r\sin\theta + \cos\nu(-bk_1 + r^2k_1) + \sin\nu(bk_2 - r^2k_2) + b\theta' - r^2\theta'\big)\Big).
\end{aligned} \tag{47}
$$

By comparing the coefficient of $\cos 5\nu$, $\sin 5\nu$ in Eq. (47), we have

$$
\begin{gathered}
a\sin\theta\,\xi_5 = 0,\\
a\ \sin\theta\big(-92r^2k_1^5 + 920r^2k_1^3k_2^2 - 460r^2k_1k_2^4\big) = 0,\\
a\sin\theta\,\eta_5 = 0,\\
a\ \sin\theta\big(460r^2k_1^4k_2 - 920r^2k_1^2k_2^3 + 92k_2^5\big) = 0.
\end{gathered}
$$

From the above equations, we have $k_1 = k_2 = 0$; this means $\kappa = 0$. Thus, the surface is a surface of revolution. From Eqs. (27) and (37), we have

$$
\begin{gathered}
2r''\big(a(r' - r'^3)^2 - r(-1 + r'^2)(-2(a+2r) + 2(a+2r)r'^2 + ar'^4)r''\\
-2r(-2b + 3ar + 4r^2)(-1 + r'^2)r''^2 - 4r^2(-b + r(a+r))r''^3\big)\\
-2arr'(-1 + r'^2)^2r^{(3)} = 0.
\end{gathered}
$$

□

Corollary 3 *The canal surface* $\Upsilon = f(u,v)$ *according to Bishop frame in* E^3 *which has non-zero second Gaussian curvature is an open part of a surface of revolution satisfies*

$$
\begin{gathered}
2r''\big(a(r' - r'^3)^2 - r(-1 + r'^2)(-2(a+2r) + 2(a+2r)r'^2 + ar'^4)r''\\
-2r(-2b + 3ar + 4r^2)(-1 + r'^2)r''^2 - 4r^2(-b + r(a+r))r''^3\big)\\
-2arr'(-1 + r'^2)^2r^{(3)} = 0.
\end{gathered}
$$

At last, we study the (x,y)*-linear Weingarten canal surface* $f(u,v)$ *according to Bishop frame in* E^3 *which satisfies* $K_{II} = K$ *and* $K_{II} = H$*, respectively.*

Theorem 10 *A non-developable canal surface* $\Upsilon = f(u,v)$ *according to Bishop frame in* E^3 *satisfying* $K_{II} = K$ *a surface of revolution which satisfies*

$$
\begin{gathered}
r'^6r''(-1 + rr'') + r'^4r''(2 + rr'') + r'^2r''\big(-1 + 4r(-1 + r'')r'' + 6r^2r''^2\big)\\
+2rr''^2\big(1 - (2 + 3r)r'' + 2r(1 + r)r''^2\big) + rr'r^{(3)} - 2rr'^3r^{(3)} + rr'^5r^{(3)} = 0,
\end{gathered}
$$

where $a, b \in R$*, and* $(a,b) \neq (0,0)$.

Proof When $K_{II} = K$, we have by Eqs. (14) and (16)

$$
\frac{1}{2048(r^2p^2q^2)} \sum_{i=0,j=-6}^{6} \sin\theta\Big(\xi_{i,j}\, \cos\left(i\nu + j\theta\right) + \eta_{i,j}\, \sin\left(i\nu + j\theta\right)\Big) = \frac{-q}{rp}. \tag{48}
$$

Comparing the coefficient of $\cos 5\nu$, $\sin 5\nu$ in Eq. (48), we have

$$
\begin{gathered}
a\sin\theta\,\xi_5 = 0,\\
a\ \sin\theta\big(-92r^2k_1^5 + 920r^2k_1^3k_2^2 - 460r^2k_1k_2^4\big) = 0,\\
a\sin\theta\,\eta_5 = 0,\\
a\ \sin\theta\big(460r^2k_1^4k_2 - 920r^2k_1^2k_2^3 + 92k_2^5\big) = 0.
\end{gathered}
$$

From the above equations, we have $k_1 = k_2 = 0$; this means $\kappa = 0$. Then, the canal surface Υ is a surface of revolution. From Eqs. (27) and (37), we have

$$r'^6 r''(-1+rr'') + r'^4 r''(2+rr'') + r'^2 r''\big(-1+4r(-1+r'')r'' + 6r^2 r''^2\big)$$
$$+2rr''^2\big(1-(2+3r)r'' + 2r(1+r)r''^2\big) + rr' r^{(3)} - 2rr'^3 r^{(3)} + rr'^5 r^{(3)} = 0,$$

□

Theorem 11 *For a non-developable canal surface $\Upsilon = f(u,v)$ according to Bishop frame in E^3 satisfying $K_{II} = H$, then the canal surface is a surface of revolution which satisfies*

$$r'^6 r''\big(-1+rr''\big) \quad +r'^4 r''\big(2+rr''\big) + 2r''^2\big(r + r'' - 3r^2 r'' - 2rr''^2 + 2r^3 r''^2\big)$$
$$-r'^2\big(r'' + 4rr''^2 + r''^3(2-6r^2)\big) + rr' r^{(3)} - 2rr'^3 r^{(3)} + rr'^5 r^{(3)} = 0.$$

Proof When $K_{II} = H$, from Eq. (19), we have

$$B = \frac{-1}{2048} \sum_{i=0,j=-6}^{6} \sin\theta\Big(\gamma_{i,j}\cos(iv + j\theta) + \delta_{i,j}\sin(iv + j\theta)\Big),$$

$$\gamma_{5,6} = \gamma_{5,-6} = r^2 k_1^5 - 10r^2 k_1^3 k_2^2 + 5r^2 k_1 k_2^4 = 0,$$
$$\delta_{2,5} = -\delta_{2,-5} = 8rk_1^2 + 8rk_1^4 + 2r^3 k_1^6 - 8rk_2^2 + 2r^3 k_1^4 k_2^2 - 8rk_2^4 - 2r^3 k_1^2 k_2^4 - 2r^3 k_2^6 = 0.$$

Considering the coefficient of $\cos(5v + 6\theta)$, $\sin(2v + 5\theta)$ in B, we get $k_1 = k_2 = 0$; this means $\kappa = 0$. Thus, the surface is a surface of revolution; we have from Eqs. (27) and (37)

$$r'^6 r''\big(-1+rr''\big) + r'^4 r''\big(2+rr''\big) + 2r''^2\big(r + r'' - 3r^2 r'' - 2rr''^2 + 2r^3 r''^2\big)$$
$$-r'^2\big(r'' + 4rr''^2 + r''^3(2-6r^2)\big) + rr' r^{(3)} - 2rr'^3 r^{(3)} + rr'^5 r^{(3)} = 0.$$

□

Conclusion

In this article, we found a particular Weingarten and linear Weingarten varieties of a canal surface obtained by the Bishop frame in Euclidean 3-space E^3 and we found the necessary and sufficient conditions of equations in terms of the Gaussian curvature, the mean curvature. On the premise of those equations, we have introduced some canal surfaces. In the future, we will try to find a comparison between the Weingarten and linear Weingarten varieties of canal surface in the Bishop frame and canal surface in Darboux frame and we will try to find the geodesic and asymptotic in Bishop frame and Darboux frame.

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