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Solvability of initial-boundary value problem of a multiple characteristic fifth-order operator-differential equation

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Abstract
In this study, we establish existence-uniqueness of a vector function in appropriate Sobolev-type space for a boundary value problem of a fifth-order operator differential equation. Proper conditions are obtained for the given problem to be well-posed. Much effort is devoted to develop the association between these conditions and the operator coefficients of the investigated equation. In this paper, accurate estimates of the norms of the intermediate derivatives operators are presented and used to determine the solvability conditions.

Keywords: Initial-boundary value problems, Operator-differential equation, Self-adjoint operator, Intermediate derivative operator

Mathematical Subject Classifications 2010: 34A12, 34G10, 34k10, 35J40, 47D03

Introduction

Initial-boundary value problem theory in Banach or Hilbert space on the real axis is useful because it enables to study equations of parabolic and elliptic differential operators with initial-boundary conditions.

We shall study the following initial-boundary value problem in a separable Hilbert space H :

$$\left(-\frac{d^2}{dx^2}+A^2\right)\left(\frac{d}{dx}+A\right)^3 u(x)+\sum_{j=1}^{5} A_j \frac{d^{5-j}}{dx^{5-j}}u(x)=f(x),$$

$$x \in R=(-\infty,+\infty) \tag{1.1}$$

$$\frac{d^s u(0)}{dx^s}=0, s=0,1,2,3 \tag{1.2}$$

where A is a self-adjoint positively defined operator and $A_j, j = 1,2,3,4,5$ are linear unbounded operators. From now on, the derivatives are accepted through the distributions theory (see [1]). We specify the following subspaces.

We consider $f(x) \in L_2(R;H)$, and $u(x) \in W_2^5(R;H)$, where:

$$L_2(R;H)=\left\{f(x):\|f(x)\|_{L_2(R;H)}=\left(\int_{-\infty}^{+\infty}\|f(x)\|_H^2\,dx\right)^{\frac{1}{2}}<+\infty\right\},$$

$W_2^5(R;H) = \left\{ u(t) : \frac{d^5u(x)}{dx^5} \in L_2(R;H), A^5u(x) \in L_2(R;H) \right\}$. With the norm $\|u\|_{W_2^5(R;H)} = \left(\| \frac{d^5u}{dx^5} \|^2_{L_2(R;H)} + \|A^5u\|^2_{L_2(R;H)} \right)^{\frac{1}{2}}$ (see [2–5]).

Definition 1 *If for any $f(x) \in L_2(R;H)$, there exists a vector function $u(x) \in W_2^5(R;H)$ that satisfies (1.1) almost everywhere in R, then it is known as a regular solution of (1.1)*

Definition 2 *If for any function $f(x) \in L_2(R;H)$, there exists a regular solution $u(x) \in W_2^5(R;H)$ of (1.1) satisfying the initial boundary conditions (1.2) in the sense that*

$$\lim_{x\to 0} \|A^{\frac{9}{2}-i} \frac{d^iu(x)}{dx^i}\|_H = 0\,, \qquad i = 0, 1, 2, 3$$

and the inequality

$$\|u\|_{W_2^5(R;H)} \leq const\|f\|_{L_2(R;H)},$$

holds, then problems (1.1) and (1.2) will be regularly solvable (see [6, 7]).

Main results

From the theorem of intermediate derivatives (see [8, 9]) if $u(x) \in W_2^5(R;H)$, then $A^{5-j}\frac{d^ju(x)}{dx^j} \in L_2(R;H), j = \overline{1,5}$ and the following inequalities:

$$\|A^{5-j}\frac{d^ju(x)}{dx^j}\|_{L_2(R;H)} \leq c_j\|u\|_{W_2^5(R;H)}, j = 1, 2, 3, 4, 5 \tag{2.1}$$

are correct.

Equation 1.1 has the following operator form: $Qu(x) \equiv Q_0u(x) + Q_1u(x) = f(x)$, where $Q_0 = \left(-\frac{d^2}{dx^2} + A^2\right)\left(\frac{d}{dx} + A\right)^3$ and $Q_1 = \sum_{j=1}^5 A_s \frac{d^{5-j}}{dx^{5-j}}$.

The following theorem provides the association between the norms of operators of intermediate derivatives and the solvability conditions of the problems (1.1) and (1.2).

Theorem 1 *The operator Q_0 isomorphically maps the space $W_2^5(R;H)$ onto the space $L_2(R;H)$, moreover, for $f(x) \in L_2(R;H)$ and Eq. 1.1 has a solution*

$$u(x) = \int_{-\infty}^{+\infty} G(x-s)f(s)ds + u_0(x),$$

where

$$G(x-s) = 2^{-4} \begin{cases} (E + 2A(x-s) + 2A^2(x-s)^2 + \\ +\frac{8}{6}A^3(x-s)^3)e^{-A(x-s)}A^{-4}, & if\ x > s, \\ e^{A(x-s)}A^{-4}, & if\ x < s, \end{cases}$$

$$\begin{aligned} u_0(x) = -2^{-4} \Bigg(& \left(\left(E + 2Ax + 2A^2x^2 + \frac{4}{3}A^3x^3\right) \int_0^\infty e^{-A(x+s)}A^{-4}f(s)ds + \right. \\ & + \left(E - 2As + 2A^2s^2 - \frac{4}{3}A^3s^3\right) \int_{-\infty}^0 e^{-A(x-s)}A^{-4}f(s)ds + \\ & + 2Ax\left(E - 2As + 2A^2s^2\right) \int_{-\infty}^0 e^{-A(x-s)}A^{-4}f(s)ds + \\ & + 2A^2x^2(E - 2As) \int_{-\infty}^0 e^{-A(x-s)}A^{-4}f(s)ds + \\ & + \frac{4}{3}A^3x^3 \int_{-\infty}^0 e^{-A(x-s)}A^{-4}f(s)ds \Bigg). \end{aligned}$$

Proof Let Eq. 1.1 has a solution $u(x) = u_1(x) + u_0(x)$, where

$$u_1(x) = \int_{-\infty}^{+\infty} G(x-s)f(s)ds,$$

and

$$\begin{aligned} u_0(x) &= \phi_0 e^{-Ax} + \phi_1 Axe^{-Ax} + \phi_2 A^2x^2e^{-Ax} + \phi_3 A^3x^3e^{-Ax},\\ \phi_0 &= -u_1(0),\\ \phi_1 &= \phi_0 - A^{-1}u_1'(0),\\ \phi_2 &= \phi_1 - \frac{1}{2}\phi_0 - \frac{1}{2}A^{-2}u_1''(0),\\ \phi_3 &= \frac{1}{6}\phi_0 - \frac{1}{6}A^{-3}u_1'''(0) - \frac{1}{2}\phi_1 + \phi_2, \end{aligned}$$

First, we find Green's function of Eq. 1.1 where, $A_j = 0,\ j = 1, 2, 3, 4, 5$.

The operator Q_0 can be simplified on the following form:

$$Q_0 = \left(-\frac{d}{dx} + A\right)\left(\frac{d}{dx} + A\right)^4,$$

then applying Fourier transform to the equation $Q_0 u(x) = f(x)$, we obtain:

$$(-i\xi E + A)(i\xi E + A)^4 \widetilde{u}(\xi) = \widetilde{f}(\xi).$$

where E- identity operator and $\widetilde{u}(\xi), \widetilde{f}(\xi)$ are Fourier transforms to the functions $u(x)$, $f(x)$, respectively.

Thus, the polynomial operator pencil $(i\xi E - A)(i\xi E + A)^4$ is invertible, and moreover,

$$\widetilde{u}(\xi) = \frac{1}{(-i\xi E + A)(i\xi E + A)^4}\widetilde{f}(\xi), \tag{2.2}$$

hence,

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-i\xi E + A)^{-1}(i\xi E + A)^{-4}\widetilde{f}(\xi)e^{i\zeta(t-s)}d\zeta.$$

Using Cauchy's theorem of residues,

at $x < s$

$$Res = \lim_{i\xi \to A} \frac{e^{i\zeta(t-s)}}{(i\xi E + A)^4} = 2^{-4}e^{A(x-s)}A^{-4},$$

at $x > s$

$$\begin{aligned} Res &= \frac{1}{6}\lim_{i\xi \to -A}\frac{d^3}{d\xi^3}\left(\frac{e^{i\zeta(t-s)}}{-i\xi E + A}\right) =\\ &= \left(E + 2A(x-s) + 2A^2(x-s)^2 + \frac{8}{6}A^3(x-s)^3\right)e^{-A(x-s)}A^{-4}. \end{aligned}$$

Then from inequality (2.1), it is simple to prove that Q_0 which acts from $W_2^5(R; H)$ to $L_2(R; H)$ is bounded (see [10]).

Now, we show that $u(x) \in W_2^5(R; H)$.

Using the Parseval's equality and (2.2), we obtain:

$$\begin{aligned} \|u\|^2_{W_2^5(R;H)} &= \|\frac{d^5u}{dx^5}\|^2_{L_2(R;H)} + \|A^5u\|^2_{L_2(R;H)} =\\ &= \|i\zeta^5\widetilde{u(\xi)}\|^2_{L_2(R;H)} + \|A^5\widetilde{u(\xi)}\|^2_{L_2(R;H)} =\\ &= \|i\zeta^5(-i\xi E + A)^{-1}(i\xi E + A)^{-4}\widetilde{f}(\xi)\|^2_{L_2(R;H)} + \end{aligned}$$

$$+\|A^5(-i\xi E+A)^{-1}(i\xi E+A)^{-4}\tilde{f}(\xi)\|^2_{L_2(R;H)}\leq$$

$$\leq \sup_{\zeta\in R}\|i\zeta^5(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|^2_{H\to H}\|\tilde{f}(\zeta)\|^2_{L_2(R;H)}+$$

$$+\sup_{\zeta\in R}\|A^5(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|^2_{H\to H}\|\tilde{f}(\zeta)\|^2_{L_2(R;H)}. \tag{2.3}$$

From the spectral decomposition of the operator A (σ(A)– the spectrum of operator A) for $\zeta \epsilon$ R (see [11]), we have:

$$\|i\zeta^5(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|_{H\to H}=\sup_{\sigma\in\sigma(A)}|i\zeta^5(-i\zeta+\sigma)^{-1}(i\zeta+\sigma)^{-4}|\leq$$

$$\leq \sup_{\sigma\in\sigma(A)}\frac{|\zeta|^5}{\left(\zeta^2+\sigma^2\right)^{\frac{5}{2}}}\leq 1, \tag{2.4}$$

$$\|A^5(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|_{H\to H}=\sup_{\sigma\in\sigma(A)}|\sigma^5(-i\zeta \mathrm{E}+\sigma)^{-1}(i\zeta \mathrm{E}+\sigma)^{-4}|\leq$$

$$\leq \sup_{\sigma\in\sigma(A)}\frac{\sigma^5}{(\zeta^2+\sigma^2)^{\frac{5}{2}}}\leq 1. \tag{2.5}$$

From (2.4) and (2.5) into (2.3), we obtain:

$$\|u\|^2_{W_2^5(R;H)}\leq 2\|\tilde{f}(\zeta)\|^2_{L_2(R;H)}=2\|f(x)\|^2_{L_2(R;H)}.$$

Hence, $u(x)\in W_2^5(R;H)$.

Using the Banach theorem of the inverse operator, then the operator Q_0 is an isomorphism from $W_2^5(R;H)$ to $L_2(R;H)$ (see [12]). □

Before we formulate exact conditions on regular solution of the problems (1.1) and (1.2), expressed only by its operator coefficients, we must estimate the norms of intermediate derivative operators participating in the perturbed part of the given equation. Theorem 1 leads to the norm $\|Q_0u\|_{L_2(R;H)}$ is equivalent to the norm $\|u\|_{W_2^5(R;H)}$ in the space $W_2^5(R;H)$. Therefore by the norm $\|Q_0u\|_{L_2(R;H)}$, the theorem on intermediate derivatives is valid as well.

Theorem 2 *When the function $u(x)\in W_2^5(R;H)$, so it keeps the following inequalities:*

$$\|A^j\frac{d^{5-j}u(x)}{dx^{5-j}}\|_{L_2(R;H)}\leq b_j\|Q_0u\|_{L_2(R;H)}, j=\overline{1,5} \tag{2.6}$$

true, where $b_1=b_4=\frac{16}{25\sqrt{5}}, b_2=b_3=\frac{6\sqrt{3}}{25\sqrt{5}}, b_5=1$ see [13].

Proof To establish the validity of inequalities (2.6), we take $Q_0u(x)=f(x)$ and apply the Fourier transformation as follows:

$$\|A^j(i\zeta)^{5-j}(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\tilde{f}(\zeta)\|_{L_2(R;H)}\leq$$

$$\leq \sup_{\zeta\in R}\|A^j(i\zeta)^{5-j}(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|_{H\to H}\|\tilde{f}(\zeta)\|_{L_2(R;H)},$$

$$j=1,2,3,4,5 \tag{2.7}$$

For $\zeta\in R$, we have:

$$\|A^j(i\zeta)^{5-j}(-i\zeta \mathrm{E}+A)^{-1}(i\zeta \mathrm{E}+A)^{-4}\|_{H\to H}\leq$$

$$\leq \sup_{\sigma\in\sigma(A)} |\sigma^j (i\zeta)^{5-j} (-i\zeta \mathrm{E} + \sigma)^{-1} (i\zeta \mathrm{E} + \sigma)^{-4}| =$$

$$= \sup_{\eta=\frac{\xi^2}{\sigma^2}\geq 0} \frac{\eta^{((5-j)/2)}}{(\eta+1)^{\frac{5}{2}}} = b_j \ , j = 1,2,3,4,5.$$

Using inequalities (2.7), we have:

$$\|A^j (i\zeta)^{5-j} (-i\zeta \mathrm{E} + A)^{-1} (i\zeta \mathrm{E} + A)^{-4} \tilde{f}(\zeta)\|_{L_2(R;H)} \leq$$

$$\leq b_j \|\tilde{f}(\xi)\|_{L_2(R;H)}, j = 1,2,3,4,5.$$

□

Lemma 1 *The operator Q_1 continuously acts from $W_2^5(R;H)$ to $L_2(R;H)$ provided that the operators $A_j A^{-j}, j = 1,2,3,4,5$ are bounded in H.*

Considering the results found up to now see [14], for problems (1.1) and (1.2), we get the possibility to establish regular solvability conditions.

Theorem 3 *Let $|\kappa| < 2\lambda_0 (A = A* \geq \lambda_o E, \lambda_o > 0)$ for any $u(t) \in W_2^5(R;H)$, then holds the inequality*

$b(k) \left(c_1(k) \|A_1 A^{-1}\|_{H\to H} + c_2(k) \|A_2 A^{-2}\|_{H\to H} + c_3(k) \|A_3 A^{-3}\|_{H\to H} + \right.$
$\left. + c_4(k) \|A_4 A^{-4}\|_{H\to H} \right) < 1$ *see* [15],

where

$$c_1(k) = \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2}\right]^{\frac{3}{2}}, \ c_2(k) = \frac{2\lambda_0}{2\lambda_0+k} \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2}\right],$$
$$c_3(k) = \frac{4\lambda_0^2}{(2\lambda_0+k)^2} \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2}\right]^{\frac{1}{2}}, \ c_4(k) = \frac{8\lambda_0^3}{(2\lambda_0+k)^3},$$

and

$$b(k) = \begin{cases} \dfrac{\lambda_0}{2^{\frac{1}{2}} (2\lambda_0^2 - k^2)^{\frac{1}{2}}}, & \text{if } \ 0 \leq \frac{k^2}{4\lambda_0^2} < \frac{1}{3}, \\ \dfrac{2\lambda_0|k|}{4\lambda_0^2 - k^2}, & \text{if } \ \frac{1}{3} \leq \frac{k^2}{4\lambda_0^2} < 1. \end{cases}$$

Theorem 4 *Suppose that the operators $A_j A^{-j}, j = \overline{1,4}$, be bounded in H and they hold the inequality*

$$\sum_{j=1}^{4} C_j(k) b(k) \|A_j A^{-j}\|_{H\to H} < 1,$$

where the numbers $C_j(k), j = 1,2,3,4$ and $b(k)$ are determined in Theorem 3 so the problems (1.1) and (1.2) are regularly solvable (see [16, 17]).

Proof where $f(x) \in L_2(R;H), u(x) \in W_2^5(R;H)$ and by Theorem (1.1), there exists a bounded inverse operator to Q_0, which acts from $L_2(R;H)$ to $W_2^5(R;H)$; then after replacing $Q_0 u(x) = w(x)$ in Eq. 1.1, it can be written as $\left(E + Q_1 Q_0^{-1}\right) w(x) = f(x)$.

Now, we prove under the theorem conditions see [18], that

$$\|Q_1 Q_0^{-1}\|_{L_2(R;H)\to L_2(R;H)} < 1.$$

By Theorem 3, we have:

$$\|Q_1 Q_0^{-1} w\|_{L_2(R;H)} = \|Q_1 u\|_{L_2(R;H)} \leq \textstyle\sum_{j=1}^{4} \|A_j \frac{d^{4-j}u}{dx^{4-j}}\|_{L_2(R;H)} \leq$$
$$\leq \textstyle\sum_{j=1}^{4} \|A_j A^{-j}\|_{H\to H} \|A^j \frac{d^{4-j}u}{dx^{4-j}}\|_{L_2(R;H)} \leq$$

$$\leq \sum_{j=1}^{4} C_j(k)b(k)\|A_jA^{-j}\|_{H\to H}\|Q_0u\|_{L_2(R;H)} =$$
$$= \sum_{j=1}^{4} C_j(k)b(k)\|A_jA^{-j}\|_{H\to H}\|v\|_{L_2(R;H)}.$$

Consequently,
$\|Q_1Q_0^{-1}\|_{L_2(R;H)\to L_2(R;H)} \leq \sum_{j=1}^{4} C_j(k)b(k)\|A_jA^{-j}\|_{H\to H} < 1$. Thus, the operator $E + Q_1Q_0^{-1}$ is invertible in $L_2(R;H)$; therefore, $u(x)$ can be determined by $u(x) = Q_0^{-1}\left(E + Q_1Q_0^{-1}\right)^{-1} f(x)$; moreover:

$$\|u\|_{W_2^5(R;H)} \leq \|Q_0^{-1}\|_{L_2(R;H)\to W_2^5(R;H)} \times \|\left(\left(E + Q_1Q_0^{-1}\right)\right)^{-1}\|_{L_2(R;H)\to L_2(R;H)}\|f\|_{L_2(R;H)}$$
$$\leq const\|f\|_{L_2(R;H)}$$

□

Conclusion

In the whole real axis, with a multiple characteristic, fifth-order and self-adjoint differential operator has not been researched so far. We demonstrated the association between the coefficients of the differential operator and the conditions of problems (1.1) and (1.2) to be regularly solvable. We estimated the norms of intermediate derivative operators which appear in the essential part of the investigated equation, and we proved that the maximum value of the $b_j, j = 1, 2, 3, 4, 5$ does not exceed one. The norms of the linear operators participating in the second part are estimated and used to formulate the exact solvability conditions.

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