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On some boundary value problems with non-local and periodic conditions

Sheren Ahmed Abd El-Salam

Correspondence:
shrnahmed@sci.dmu.edu.eg;
shrnahmed@yahoo.com
Faculty of Science, Damanhour University, Damanhour, Egypt

Abstract
In this work, we concern non-local and periodic boundary value problems. We will prove the existence of at least one solution of these problems such that the functions satisfy the growth condition. Hence, we will study the existence of at least one solution for a boundary value problem with periodic and integrable conditions.

Keywords: Non linear boundary value problems, Non-local boundary value problems, Periodic conditions

AMS Subject classification: 34B15; 34B10; 34C25

Introduction

Differential equations with non-local conditions were considered in many works (see [1], [2], [3], and [4]). Also, anti-periodic problems can be found in [5] and [6].

Here, we study the existence of at least one solution for the boundary value problem with non-local and periodic conditions:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \quad \text{a.e. } t \in (0, 2\pi), \\ x(0) = x(2\pi) \text{ and } \sum_{k=1}^{m} a_k \, x(\tau_k) = x_0 \end{cases} \tag{1}$$

where $x_0 \in R$, $0 < \tau_1 < \tau_2 < \cdots < \tau_m < 2\pi$ and $a_k \neq 0$ for all $k = 1, 2, \cdots, m$.

Also, the boundary value problem with integral and periodic conditions:

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) \quad \text{a.e. } t \in (0, 2\pi), \\ x(0) = x(2\pi) \text{ and } \int_0^{2\pi} x(t)\, dt = x_0 \end{cases} \tag{2}$$

will be considered.

Problem (2) was studied in [7], but the author has not shown the equivalence between the differential problem (2) and the integral equation equivalent with it.

Here, we prove, by using nonlinear alternative of Leray-Schauder type, the existence of at least one solution for problem (1) such that the function $f : I \times R \times R \to R, I = [0, 2\pi]$ satisfies the growth conditions.

Preliminaries

Theorem 1 ***(Nonlinear alternative of Leray-Schauder type)*** *[8] Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$ and $T : \bar{\Omega} \to E$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega, \lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \bar{\Omega}$.*

Denote by $C(I)$ the space of all continuous functions defined on the interval I with norm

$$||u||_C = \sup_{t\in I} |u(t)|$$

and by $L_1(I)$ the space of all Lebesgue integrable functions on the interval I with norm

$$||u||_{L_1} = \int_I |u(t)|\, dt.$$

The growth condition on the function f means that

$$|f(t,u)| \le a(t) + b\,|u|,$$

where $a(t) \in L_1$, b is a nonnegative constant.

Main results

Let the function $f : I \times R \times R \to R$ satisfy the following assumptions:

(1) $f : I \times R \times R \to R$ is measurable in $t \in I$ for any $(u_1, u_2) \in R \times R$
(2) $f : I \times R \times R \to R$ is continuous in $(u_1, u_2) \in R \times R$ for any $t \in I$
(3) There exist two positive constants b_1, b_2 and a function $c(t) \in L_1(I)$ such that

$$|f(t,\ u_1,\ u_2)| \le c(t) + b_1\,|u_1| + b_2\,|u_2|.$$

Integral representation

Lemma 1 *Let the assumptions (1)–(3) be satisfied. If the solution of the boundary value problem (1) exists, then it can be represented by*

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) + \left(t - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^t (t - s)\, y(s)\, ds,$$

where

$$\begin{aligned} y(t) &= f\left(t, y_1(t), y_2(t)\right), \\ y_1(t) &= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\ &\quad + \left(t - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) \\ &\quad + \int_0^t (t - s)\, y(s)\, ds \\ \text{and}\quad y_2(t) &= \frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^t y(s)\, ds, \quad t \in I. \end{aligned} \tag{3}$$

Proof Let $y = x''(t) = f(t, x, x')$. □

Integrating both sides, we obtain

$$x'(t) - x'(0) = \int_0^t y(s)\, ds.$$

Integrating again, we get

$$\begin{aligned} x(t) &= x(0) + t\,x'(0) + \int_0^t \int_0^s y(\theta)\, d\theta\, ds \\ &= x(0) + t\,x'(0) + \int_0^t (t - s)\, y(s)\, ds. \end{aligned}$$

From the boundary condition, we obtain

$$x'(0) = \frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\, y(s)\, ds,$$

then

$$\begin{aligned} x'(t) &= x'(0) + \int_0^t y(s)\, ds \\ &= \frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^t y(s)\, ds. \end{aligned} \tag{4}$$

Now,

$$x(t) = x(0) + t\,x'(0) + \int_0^t (t - s)\, y(s)\, ds,$$

$$\text{then} \quad x(\tau_k) = x(0) + \tau_k\, x'(0) + \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds$$

$$\text{and} \quad \sum_{k=1}^m a_k\, x(\tau_k) = x(0) \sum_{k=1}^m a_k + \sum_{k=1}^m a_k\, \tau_k\, x'(0) + \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds.$$

Take $A = (\sum_{k=1}^m a_k)^{-1}$, then

$$x(0) = A\left(x_0 - \sum_{k=1}^m a_k\, \tau_k\, x'(0) - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right).$$

Substituting the values of $x'(0)$ and $x(0)$ in $x(t)$, we get

$$\begin{aligned} x(t) &= A\left(x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\ &\quad + \left(t - A \sum_{k=1}^m a_k\, \tau_k\right) \left(\frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^t (t - s)\, y(s)\, ds. \end{aligned} \tag{5}$$

Inserting (4) and (5) in $x''(t) = f(t, x(t), x'(t))$, we get

$$\begin{aligned} y(t) &= f\,(t, y_1(t), y_2(t)) \\ &= f\left(t,\, A\left(x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right)\right. \\ &\quad + \left(t - A \sum_{k=1}^m a_k\, \tau_k\right) \left(\frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) \\ &\quad \left. + \int_0^t (t - s)\, y(s)\, ds,\, \frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^t y(s)\, ds\right), \quad t \in [0, 2\pi]. \end{aligned}$$

Existence of solution

Define the operator T by

$$T\,y(t) = f\,(t,\, y_1(t),\, y_2(t))\,,\ \ t \in I$$

where

$$\begin{aligned} y_1(t) &= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\ &+ \left(t - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) \\ &+ \int_0^t (t - s)\, y(s)\, ds \end{aligned}$$

and

$$y_2(t) = \frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^t y(s)\, ds.$$

Firstly, we prove that the functional Eq. (3) has at least one solution $y \in L_1(I)$; in order to do that, we will show that the operator T has a fixed point $y \in L_1(I)$.

Theorem 2 *Let the function $f : I \times R \times R \to R$ satisfy the assumptions (1)–(3) and the following assumption:*

(4) Every solution $y(.) \in L_1(I)$ to the equation

$$y(t) = \gamma\, f\,(t,\, y_1(t),\, y_2(t))\ \ \text{a.e. on } I,\ \gamma \in (0, 1)$$

satisfies $||y||_{L_1} \neq r$ (r is arbitrary but fixed).

Then the operator T has a fixed point $y \in L_1(I)$, which is a solution to Eq. (3).

Proof Let y be an arbitrary element in the open set $B_r = \{y : ||y||_{L_1} < r,$ $r = \frac{||c||_{L_1} + 2\pi b_1 |A||x_0|}{1-(8\pi^2 b_1 + 2\pi\ b_1\ |A|\ |\sum_{k=1}^{m}\ a_k\ \tau_k| + 4\ \pi\ b_2)} > 0\}$. Then from the assumptions (1)–(3), we have

$$
\begin{aligned}
||Ty||_{L_1} &= \int_0^{2\pi} |Ty(t)|\ dt \\
&= \int_0^{2\pi} \left| f\left(t, y_1(t), y_2(t)\right) \right|\ dt \\
&\le \int_0^{2\pi} \left[|c(t)| + b_1\ |y_1(t)| + b_2\ |y_2(t)| \right]\ dt \\
&\le ||c||_{L_1} + b_1 \int_0^{2\pi} |y_1(t)|\ dt + b_2 \int_0^{2\pi} |y_2(t)|\ dt \\
&\le ||c||_{L_1} + b_1 \int_0^{2\pi} \left| A \left(x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds \right) \right. \\
&+ \left(t - A \sum_{k=1}^m a_k\ \tau_k \right) \left(\frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right) \\
&+ \left. \int_0^t (t - s)\ y(s)\ ds \right| dt + b_2 \int_0^{2\pi} \left| \frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds + \int_0^t y(s)\ ds \right| dt \\
&\le ||c||_{L_1} + b_1 \int_0^{2\pi} \left| A \left(x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds \right) \right| dt \\
&+ b_1 \int_0^{2\pi} \left| \left(t - A \sum_{k=1}^m a_k\ \tau_k \right) \left(\frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right) \right| dt \\
&+ b_1 \int_0^{2\pi} \left| \int_0^t (t - s)\ y(s)\ ds \right| dt + b_2 \int_0^{2\pi} \left| \frac{-1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right| dt \\
&+ b_2 \int_0^{2\pi} \int_0^t |y(s)|\ ds\ dt \\
&\le ||c||_{L_1} + b_1 \int_0^{2\pi} |A\ x_0|\ dt + b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k \int_0^{\tau_k} (\tau_k - s)\ y(s)\ ds \right| dt \\
&+ b_1 \int_0^{2\pi} t \left| \frac{1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right| dt \\
&+ b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k\ \tau_k \right| \left| \frac{1}{2\pi} \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right| dt \\
&+ b_1 \int_0^{2\pi} \int_s^{2\pi} (t - s)\ |y(s)|\ dt\ ds + b_2 \int_0^{2\pi} \int_0^{2\pi} (1 - \frac{s}{2\pi})\ |y(s)|\ ds\ dt \\
&+ b_2 \int_0^{2\pi} \int_s^{2\pi} |y(s)|\ dt\ ds \\
&\le ||c||_{L_1} + 2\pi\ b_1\ |A|\ |x_0| + b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k \int_0^{2\pi} (2\pi - s)\ y(s)\ ds \right| dt \\
&+ b_1 \int_0^{2\pi} t \int_0^{2\pi} (1 - \frac{s}{2\pi})\ |y(s)|\ ds\ dt + b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k\ \tau_k \right| \int_0^{2\pi} (1 - \frac{s}{2\pi})\ |y(s)|\ ds\ dt \\
&+ b_1 \int_0^{2\pi} \frac{(t - s)^2}{2} \Big|_s^{2\pi} |y(s)|\ ds + b_2 \int_0^{2\pi} \int_0^{2\pi} |y(s)|\ ds\ dt \\
&+ b_2 \int_0^{2\pi} (2\pi - s)\ |y(s)|\ ds \\
&\le ||c||_{L_1} + 2\pi\ b_1\ |A|\ |x_0| + b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k \right| \int_0^{2\pi} (2\pi)\ |y(s)|\ ds\ dt \\
&+ b_1 \int_0^{2\pi} t \int_0^{2\pi} |y(s)|\ ds\ dt + b_1 \int_0^{2\pi} \left| A \sum_{k=1}^m a_k\ \tau_k \right| \int_0^{2\pi} |y(s)|\ ds\ dt \\
&+ b_1 \int_0^{2\pi} \frac{(2\pi - s)^2}{2}\ |y(s)|\ ds + b_2\ ||y||_{L_1} \int_0^{2\pi} dt + 2\pi\ b_2 \int_0^{2\pi} |y(s)|\ ds \\
&\le ||c||_{L_1} + 2\pi\ b_1\ |A|\ |x_0| + (2\pi)^2\ b_1\ ||y||_{L_1} + b_1\ ||y||_{L_1} \frac{(2\pi)^2}{2} \\
&+ 2\pi\ b_1\ |A|\ |\sum_{k=1}^m a_k\ \tau_k|\ ||y||_{L_1} + \frac{(2\pi)^2}{2}\ b_1\ ||y||_{L_1} + 2\pi\ b_2\ ||y||_{L_1} + 2\pi\ b_2\ ||y||_{L_1} \\
&\le ||c||_{L_1} + 2\pi\ b_1\ |A|\ |x_0| + 4\ \pi^2\ b_1\ ||y||_{L_1} + 2\pi^2\ b_1\ ||y||_{L_1} \\
&+ 2\pi\ b_1\ |A|\ |\sum_{k=1}^m a_k\ \tau_k|\ ||y||_{L_1} + 2\pi^2\ b_1\ ||y||_{L_1} + 4\ \pi\ b_2\ ||y||_{L_1} \\
&= ||c||_{L_1} + 2\pi\ b_1\ |A|\ |x_0| + 8\ \pi^2\ b_1\ ||y||_{L_1} + 2\pi\ b_1\ |A|\ |\sum_{k=1}^m a_k\ \tau_k|\ ||y||_{L_1} + 4\ \pi\ b_2\ ||y||_{L_1}.
\end{aligned}
$$

The above inequality means that the operator T maps B_r into L_1. □

Also, from assumption (2), we deduce that T maps B_r continuously into $L_1(I)$.

Now, we will use Kolmogorov compactness criterion (see [9]) to show that T is compact. So, let $\aleph$ be a bounded subset of B_r. Then $T(\aleph)$ is bounded in $L_1(I)$. Now we show that $(Ty)_h \to Ty$ in $L_1(I)$ as $h \to 0$, uniformly with respect to $Ty \in T\,\aleph$.

Indeed:

$$\begin{aligned}||(Ty)_h - Ty||_{L_1} &= \int_0^{2\pi} |\,(Ty)_h(t) - (Ty)(t)\,|\,dt \\ &= \int_0^{2\pi} \left|\frac{1}{h}\int_t^{t+h} (Ty)(s)\,ds - (Ty)(t)\right| dt \\ &\le \int_0^{2\pi} \left(\frac{1}{h}\int_t^{t+h} |\,(Ty)(s) - (Ty)(t)\,|\,ds\right) dt \\ &\le \int_0^{2\pi} \frac{1}{h}\int_t^{t+h} |f(s, y_1(s), y_2(s)) - f(t, y_1(t), y_2(t))\,|\,ds\,dt.\end{aligned}$$

Since

$$||f||_{L_1} \le ||c||_{L_1} + 2\pi\, b_1\, |A|\, |x_0| + 8\,\pi^2\, b_1\, ||y||_{L_1} + 2\pi\, b_1\, |A|\, |\sum_{k=1}^{m} a_k\, \tau_k|\, ||y||_{L_1} + 4\,\pi\, b_2\, ||y||_{L_1},$$

we have that f in $L_1(I)$. So, we have (see [10])

$$\frac{1}{h}\int_t^{t+h} |f(s, y_1(s), y_2(s)) - f(t, y_1(t), y_2(t))\,|\,ds \to 0,$$

for a.e. $t \in I$. So, $T(\aleph)$ is relatively compact, that is, T is a compact operator.

Now from assumption (4) and Theorem 1, we get that T has a fixed point $y \in L_1(I)$.

Theorem 3 *If the assumptions of Theorem 2 are satisfied, then the periodic and non-local boundary value problem (1) has at least one solution $x \in C^1(I)$.*

Proof Let $x(t)$ be a solution of (5)

$$\begin{aligned}x(t) &= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\ &\quad + \left(t - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^t (t - s)\, y(s)\, ds,\end{aligned}$$

by differentiation, we obtain

$$x'(t) = \frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^t y(s)\, ds.$$

Since Theorem 2 proved that $y \in L_1(I)$, then by differentiating again, we get

$$x''(t) = y(t) = f(t, x(t), x'(t)).$$

Substituting respectively by $x = 0$ and $x = 2\pi$ in (5), we get

$$x(0) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) + \left(-A\sum_{k=1}^{m} a_k \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) \tag{6}$$

and

$$
\begin{aligned}
x(2\pi) &= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\
&+ \left(2\pi - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^{2\pi} (2\pi - s)\, y(s)\, ds \\
&= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) + \left(-A\sum_{k=1}^{m} a_k \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right).
\end{aligned} \tag{7}
$$

From (6) and (7), we get $x(0) = x(2\pi)$. □

Also,

$$
\begin{aligned}
x(\tau_k) &= A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\
&+ \left(\tau_k - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds, \\
a_k\, x(\tau_k) &= a_k\, A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) \\
&+ a_k\left(\tau_k - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s) y(s)\, ds\right) + a_k \int_0^{\tau_k} (\tau_k - s) y(s)\, ds, \\
\sum_{k=1}^{m} a_k\, x(\tau_k) &= \sum_{k=1}^{m} a_k\, A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) y(s)\, ds\right) \\
&+ \sum_{k=1}^{m} a_k\left(\tau_k - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s) y(s)\, ds\right) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s) y(s)\, ds \\
&= x_0\, .
\end{aligned}
$$

Then the periodic and non-local boundary value problem (1) is equivalent to the integral Eq. (5). Hence problem (1) has at least one solution $x \in C^1(I)$.

Theorem 4 *If $f : I \times R \times R \to R$ satisfies the assumptions of Theorem 2, then the boundary value problem (2) has at least one solution $x \in C^1(I)$, and its solution is given by*

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi}\left(x_0 - \int_0^{2\pi} \frac{(2\pi - s)^2}{2}\, y(s)\, ds\right) \\
&+ (t - \pi)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^{t} (t - s)\, y(s)\, ds.
\end{aligned} \tag{8}
$$

Also,

$$
\begin{aligned}
x'(t) &= x'(0) + \int_0^{t} y(s)\, ds \\
&= \frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds + \int_0^{t} y(s)\, ds.
\end{aligned}
$$

Proof If we take $a_k = t_k - t_{k-1}, \tau_k \in (t_{k-1}, t_k)$ and $0 < t_1 < t_2 < \ldots < 2\pi$, we get

$$
\sum_{k=1}^{m} (t_k - t_{k-1}) x(\tau_k) = x_0.
$$

By taking the limit as $m \to \infty$, we get $\int_0^{2\pi} x(t)dt = x_0$. □

Also, take the limit as $m \to \infty$ in (5):

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} (\tau_k - s)\, y(s)\, ds\right) + \left(t - A\sum_{k=1}^{m} a_k\, \tau_k\right)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^t (t - s)\, y(s)\, ds,$$

we obtain (8):

$$x(t) = \frac{1}{2\pi}\left(x_0 - \int_0^{2\pi} \frac{(2\pi - s)^2}{2}\, y(s)\, ds\right) + (t - \pi)\left(\frac{-1}{2\pi}\int_0^{2\pi} (2\pi - s)\, y(s)\, ds\right) + \int_0^t (t - s)\, y(s)\, ds.$$

This completes the proof.

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