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Some fixed point results in fuzzy cone normed linear space

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Abstract

In this paper, the well known fixed point theorems of Banach, Kannan, and Chatterjee are extended to the fuzzy cone normed linear space.

Keywords: Fuzzy cone norm, Strongly minihedral cone, α-convergent, α- Cauchy

Mathematics subject classification: 46S40, 03E72, 32C25

Introduction

The concept of fuzzy norm was first introduced by Katsaras [1] in the year 1984. After that, in 1992, Felbin [2] defined a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [3]. Further developement in the notion of fuzzy norm took place in 1994, when Cheng and Moderson [4] gave the definition of fuzzy norm in another approach having an associated metric of the Kramosil and Michalek type [5] . Thereafter, following the definition of fuzzy norm by Cheng and Moderson [4], Bag and Samanta [6] introduced the concept of fuzzy norm in a different way.

On the other hand, several authors generalized the concept of metric space in many ways. One of them is the notion of cone metric space introduced by Long-Guang et al. [7] in the year 2007. In the year 2017, Tamang and Bag [8] extended the concept of fuzzy norm to fuzzy cone norm with replacement of R by a real Banach space. In 1922, Banach [9] proved fixed point result on contractive type mappings. So far, many authors have obtained interesting extensions and generalization of the Banach contraction principle. In 1968, Kannan [10] and, in 1972, Chatterjee [11] studied contractive mappings which gives unique fixed point on complete metric space. As the fuzzy mathematics along with the classical ones are constantly developing, the above fixed point results in fuzzy cone normed linear space setting can also play an important role. Our aim in this paper is to establish Banach, Kannan, and Chatterjee type fixed point theorems in fuzzy cone normed linear space setting.

Preliminaries

In this section, some essential concepts for study are stated. Throughout the paper we use symbol $\bigwedge$ to denote the infimum.

Definition 1 [7] *Let E be a real Banach space and P be a subset of E. P is called a cone if and only if:*

(i) P is closed, nonempty and $P \neq \{\theta\}$;
(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \ \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ iff $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$ while $x << y$ will stand for $y - x \in$ Int$P$, where Int$P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, with $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$.

The least positive number satisfying above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence in E such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Equivalently, the cone P is regular if every decreasing sequence is bounded below is convergent. It is clear that a regular cone is a normal cone.

Definition 2 [12] *The cone P is called strongly minihedral if every subset of E which is bounded above via the partial ordering obtained by P, must have a least upper bound. Hence, every subset which is bounded below must have greatest lower bound.*

Definition 3 [13] *A binary operation* $* : [0, 1] \times [0, 1] \to [0, 1]$ *is a t-norm if it satisfies the following conditions:*

(1) $*$ is associative and commutative;
(2) $a * 1 = a \ \ \forall a \in [0, 1]$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

If $*$ is continuous then it is called continuous t-norm. The following are examples of some t-norms that are frequently used and defined for all $a,\ b \in [0,\ 1]$.

(i) Standard intersection: $a * b = min(a,\ b)$.
(ii) Algebraic product: $a * b = ab$.
(iii) Bounded difference: $a * b = max(0,\ a + b - 1)$.
(iv) Drastic intersection:

$$a * b = \begin{cases} a & \text{for } b = 1 \\ b & \text{for } a = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4 [8] *Let X be a linear space over the field K and E be a real Banach space with cone P,* $*$ *is a t-norm. A fuzzy subset* $N_c : X \times E \longrightarrow [0, 1]$ *is said to be a fuzzy cone norm if*

($FCN1$) $\forall\, t \in E$ with $t \preceq \theta_E,\ N_c(x, t) = 0$;
($FCN2$) $(\forall\, \theta_E \prec t,\ N_c(x, t) = 1)$ iff $x = \theta_X$;
($FCN3$) $\forall\, \theta_E \prec t$ and $0 \neq c \in K,\ N_c(cx, t) = N_c\left(x, \frac{t}{|c|}\right)$;
($FCN4$) $\forall\, x, y \in X$ and $s, t \in E,\ \ N_c(x + y, s + t) \geq N_c(x, s) * N_c(y, t)$;
($FCN5$) $\lim_{\|t\| \to \infty} N_c(x, t) = 1$;

Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t E.

Theorem 1 (Banach [9]) *Let f be a self-map of a complete metric space* (X, d) *such that* $d(f(x), f(y)) \leq \alpha d(x, y)$ *for some real number* α, $0 < \alpha < 1$ *for each* $x, y \in X$. *Then f has a unique fixed point.*

Theorem 2 (Kannan [10]) *Let f be a self-map of a complete metric space* (X, d) *such that* $d(f(x), f(y)) \leq \beta[d(f(x), x) + d(f(y), y)]$ *for some real number* $\beta, 0 < \beta < \frac{1}{2}$ *for each* $x, y \in X$. *Then f has a unique fixed point.*

Theorem 3 (Chatterjee [11]) *Let f be a self-map of a complete metric space* (X, d) *such that* $d(f(x), f(y)) \leq \gamma[d(f(x), y) + d(f(y), x)]$ *for some real number* $\gamma, 0 < \gamma < \frac{1}{2}$ *for each* $x, y \in X$. *Then f has a unique fixed point.*

Main results

In this section we modified Definition 4 of fuzzy cone normed linear space as follows in order to develop some fixed point results.

Definition 5 *Let X be a linear space over the field K and E be a real Banach space with cone P,* $*$ *is a t-norm. A fuzzy subset* $N_c : X \times E \longrightarrow [0, 1]$ *is said to be a fuzzy cone norm if*

(FCN1) $\forall\, t \in E$ with $t \preceq \theta_E$, $N_c(x, t) = 0$;
(FCN2) $(\forall\, \theta_E \prec t,\ N_c(x, t) = 1)$ iff $x = \theta_X$;
(θ_X denotes the zero element of X)
(FCN3) $\forall\, \theta_E \prec t$ and $0 \neq c \in K$, $N_c(cx, t) = N_c\left(x, \frac{t}{|c|}\right)$;
(FCN4) $\forall\, x, y \in X$ and $s, t \in E$, $N_c(x + y, s + t) \geq N_c(x, s) * N_c(y, t)$;
(FCN5) $N_c(x, t) = 1$ if $s \prec t\ \forall\, s \in P$;
Then $(X, N_c, *)$ is said to be a fuzzy cone normed linear space w.r.t. E.
(FCN6) $N_c(x, t) > 0\ \forall\, t \succ \theta_E \Rightarrow x = \theta_X$.

Note. We notice that for a real Banach space with normal cone the modified definition is stronger than the existing one.

For, $s \prec t\ \forall s \in P$
i.e, $\theta_E \preceq s \prec t\ \forall\, s \in P$
$\Rightarrow \|s\| \leq K\|t\|\ \forall s \in P$ (K is a normal constant)
$\Rightarrow \|t\| >$ any positive real number
$\Rightarrow \|t\| \longrightarrow \infty$.

Definition 6 *Let* $(X, N_c, *)$ *be a fuzzy cone normed linear space with a strongly minihedral cone P and* $\alpha \in (0, 1)$. *A sequence* $\{x_n\}$ *is said to be* α*-fuzzy convergent and converges to x if* $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$. *If* $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E \quad \forall\alpha \in (0, 1)$, *then* $\{x_n\}$ *is said to be l-fuzzy convergent and converges to x.*

Definition 7 *Let* $(X, N_c, *)$ *be a fuzzy cone normed linear space with a strongly minihedral cone P and* $\alpha \in (0, 1)$. *A sequence* $\{x_n\}$ *is said to be* α*-fuzzy Cauchy sequence if* $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E$. *for each* $p = 1, 2, 3, \ldots$
If $\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} = \theta_E \quad \forall\alpha \in (0, 1)$ *and for each* $p = 1, 2, 3, \ldots$, *then* $\{x_n\}$ *is said to be l-fuzzy Cauchy sequence.*

Definition 8 *Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0, 1)$. Then, X is said to be α-fuzzy complete if every α-fuzzy Cauchy sequence is α-fuzzy convergent to some element in X.*

Definition 9 *Let $(X, N_c, *)$ be a fuzzy cone normed linear space with a strongly minihedral cone P and $\alpha \in (0, 1)$. Then X is said to be l-fuzzy complete if every α-fuzzy Cauchy sequence is α-fuzzy convergent $\forall \alpha \in (0, 1)$.*

Example 1 *Let $(X, \| \|_c)$ be a cone normed linear space and take $E = R^2$. Then $P = \{(t_1, t_2) : t_1, t_2 \geq 0\} \subset E$ is a strongly minihedral normal cone with normal constant 1. Define a function $N_c : X \times E \longrightarrow [0, 1]$ by*

$$\begin{aligned} N_c(x, t) &= 1 \quad \text{if } \|x\|_c \prec t \\ &= 0 \quad \text{if } t \preceq \|x\|_c \end{aligned}$$

If we choose $* = min$, Then $(X, N_c, *)$ is a fuzzy cone normed linear space satisfying $(FCN6)$. If we take $X = R$, then $(X, N_c, *)$ is an l-fuzzy complete fuzzy cone normed linear space

Proof:

(i) $\forall\, t \in E$ with $t \preceq \theta_E$, we have by definition, $N_c(x, t) = 0$ for all $x \in X$. Thus $(FCN1)$ holds.

(ii) $\forall\, t \in E$ with $\theta_E \prec t$,
$N_c(x, t) = 1$
$\Rightarrow \|x\|_c \prec t\ \forall t \succ \theta_E$
$\Rightarrow \| \|x\|_c\| \leq \|t\|\ \forall t \succ \theta_E$ (since P is normal cone with normal constant 1)
$\Rightarrow \| \|x\|_c\| = 0.$
$\Rightarrow \|x\|_c = \theta_E.$
$\Rightarrow x = \theta_X$ (θ_X denotes the zero element of X)
Again $x = \theta_X$
$\Rightarrow \|x\|_c = \theta_E.$
$\Rightarrow \|\theta_X\|_c \prec t\ \forall t \succ \theta_E$
$\Rightarrow N_c(x, t) = 1.$
So $(FCN2)$ holds.

(iii) For all $t \in E$ with $\theta_E \prec t$ and $0 \neq c \in K$
Let $N_c(cx, t) = 0$
$\Rightarrow t \preceq \|cx\|_c$
$\Rightarrow t \preceq |c|\|x\|_c$
$\Rightarrow \frac{t}{|c|} \preceq \|x\|_c \Rightarrow N_c\left(x, \frac{t}{|c|}\right) = 0.$
Let $N_c(cx, t) = 1$
$\Rightarrow \|cx\|_c \preceq t$
$\Rightarrow |c|\|x\|_c \preceq t$
$\Rightarrow \|x\|_c \preceq \frac{t}{|c|}$
$\Rightarrow N_c\left(x, \frac{t}{|c|}\right) = 1.$
So $(FCN3)$ holds.

(iv) We have to show that
$N_c(x + u, s + t) \geq min\{N_c(x, s), N_c(u, t)\}\ \ \forall x, y \in X$ and $s, t \in E$

If $N_c(x+u, s+t) = 0$
Then, $s + t \preceq \|x + u\|_c \preceq \|x\|_c + \|u\|_c$

$$\Rightarrow \|x\|_c + \|u\|_c - (s+t) \in P$$
$$\Rightarrow \|u\|_c - t - (s - \|x\|_c) \in P$$
$$\Rightarrow s - \|x\|_c \preceq \|u\|_c - t \tag{1}$$

If $\|x\|_c \prec s$ i.e, $\theta_E \prec s - \|x\|_c$, then from (1)
$\theta_E \prec \|u\|_c - t$
$\Rightarrow t \prec \|u\|_c$
So if $\|x\|_c \prec s$, then $t \prec \|u\|_c$
So $N_c(x, s) = 1$ and $N_c(u, t) = 0$.
Similarly, if $\|u\|_c \prec t$, then $s \prec \|x\|_c$
So $N_c(u, t) = 1$ and $N_c(x, s) = 0$.
So in both cases,
$N_c(x+u, s+t) \geq min\{N_c(x, s), N_c(u, t)\} = 0$.
If $N_c(x+u, s+t) = 1$
Then $N_c(x+u, s+t) \geq min\{N_c(x, s), N_c(u, t)\}$
So $(FCN4)$ holds.

(v) If $s \prec t$ for every $s \in P$, then by definition, $N_c(x, t) = 1$. So $(FCN5)$ holds.
Thus, $(X, N_c, *)$ is a fuzzy cone normed linear space.
Now, $\forall t \succ \theta_E,\ \ N_c(x, t) > 0$
$\Rightarrow N_c(x, t) = 1,\ \forall t \succ \theta_E$
$\Rightarrow \|x\|_c \prec t\ \forall t \succ \theta_E$
$\Rightarrow \|\|x\|_c\| \leq \|t\|\ \forall t \succ \theta_E$ (P is a normal cone with normal constant 1)
$\Rightarrow \|x\|_c = \theta_E$
$\Rightarrow x = \theta_X$.
Thus, $(FCN6)$ holds.
We now prove that $(X, N_c, *)$ is an l-fuzzy complete cone normed linear space.
Let $\{x_n\}$ be a α-Cauchy sequence in $(X, N_c, *)$ for $\alpha \in (0, 1)$.
Then $\bigwedge\{t \succ \theta_E : N_c(x_n - x_m, t) \geq \alpha\} = \theta_E$ as $m, n \to \infty$
Choose $\epsilon \succ \theta_E$ arbitrarily, then there exists a natural number p such that
$\bigwedge\{t \succ \theta_E : N_c(x_n - x_m, t) \geq \alpha\} \prec \epsilon\ \forall t \succ \theta_E$ and $m, n \geq p$.
$\Rightarrow N_c(x_n - x_m, \epsilon) \geq \alpha > 0\ \forall \epsilon \succ \theta_E$ and $m, n \geq p$.
$\Rightarrow \|x_n - x_m\|_c \prec \epsilon\ \forall \epsilon \succ \theta_E$ and $m, n \geq p$. (by the definition of N_c)
$\Rightarrow \|\|x_n - x_m\|_c\| \leq \|\epsilon\|\ \forall \epsilon \succ \theta_E$ (since P is normal cone with normal constant 1)
$\Rightarrow \|x_n - x_m\|_c \to \theta_E$ as $m, n \to \infty$
$\Rightarrow |x_n - x_m| \to 0$ as $m, n \to \infty$
$\Rightarrow \{x_n\}$ is a Cauchy sequence in R. Since R is complete, $\exists\, x \in R$ such that $x_n \to x$ as $n \to \infty$
$\Rightarrow x_n - x \to 0$ as $n \to \infty$
$\Rightarrow \|x_n - x\|_c \to \theta_E$ as $n \to \infty$
$\Rightarrow$ there exists a natural number $n_0(t)$ such that $\|x_n - x\|_c \prec t\ \forall\ t \succ \theta_E$ and $n \geq n_0(t)$.
$\Rightarrow N_c(x_n - x, t) = 1\ \forall\ t \succ \theta_E$ and $n \geq n_0(t)$.
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E$ as $n \to \infty$
$\Rightarrow \{x_n\}$ is α-convergent to x.

Since $\alpha \in (0,1)$ is arbitrary, every α-Cauchy sequence is α-convergent. So $(X, N_c, *)$ is an I-fuzzy complete fuzzy cone normed linear space.

Some fixed point theorems

In this section Banach, Kannan, and Chatterjee type fixed point theorems are established in fuzzy setting.

Throughout this section, we consider $*$ as continuous t-norm.

Theorem 4 (Banach Contraction type theorem in fuzzy cone normed linear space)

Let $(X, N_c, *)$ be an l-fuzzy complete cone normed linear space satisfying $(FCN6)$, P be a stronghly minihedral cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition

$$\bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}$$

for some, $\alpha \in (0,1)$ and $k \in (0,1)$ is a constant. Then, T has a fixed point in X. In addition if $M = 1$, then the fixed point is unique. Assuming that $\beta * \beta > 0 \; \forall \, \beta \in (0,1)$.

Proof Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$. So $\{x_n\}$ is a sequence in X. First, we show that $\{x_n\}$ is β-Cauchy sequence for some $\beta \in (0,1)$. □

Now, for some $\alpha \in (0,1)$;

$$\begin{aligned}\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} &= \bigwedge\{t \succ \theta_E : N_c(Tx_n - Tx_{n-1}, t) \geq \alpha\} \\ &\preceq k \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\}\end{aligned}$$

i.e., $\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq k \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\}$

Proceeding similarly, we have

$$\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq k^n \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\}$$

Using normality condition, we get

$$\begin{aligned}&\| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\| \leq k^n M \| \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\}\| \\ &\Rightarrow \lim_{n\to\infty} \| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}\| = 0 \;\; (\text{since } k^n \to 0 \text{as } n \to \infty) \\ &\Rightarrow \lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} = \theta_E \qquad (2)\end{aligned}$$

Now for $p \geq 1$, we have

$$\begin{aligned}&\bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+p} - x_{n+p-1}, \frac{t}{p}\right) \geq \alpha\right\} \\ &+ \bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}\right) \geq \alpha\right\} \\ &+ ... + \bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+1} - x_n, \frac{t}{p}\right) \geq \alpha\right\} \\ &\succeq \bigwedge\left\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * ... * \alpha\right\}\end{aligned}$$

Since $*$ is continuous, $\exists \beta \in (0,1)$ such that $\alpha * \alpha * ... * \alpha = \beta$.

From above, we get

$$p\bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_{n+p-1},t)\geq\alpha\}+p\bigwedge\{t\succ\theta_E : N_c(x_{n+p-1}-x_{n+p-2},t)\geq\alpha\}$$
$$+\ldots+p\bigwedge\{t\succ\theta_E : N_c(x_{n+1}-x_n,t)\geq\alpha\}\succeq\bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_n,t)\geq\beta\}$$

Using normality condition, we have,

$$\left\|\bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_n,t)\geq\beta\}\right\|\leq pM\left\|\bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_{n+p-1},t)\geq\alpha\}\right\|$$
$$+pM\left\|\bigwedge\{t\succ\theta_E : N_c(x_{n+p-1}-x_{n+p-2},t)\geq\alpha\}\right\|+\ldots$$
$$+pM\left\|\bigwedge\{t\succ\theta_E : N_c(x_{n+1}-x_n,t)\geq\alpha\}\right\|$$

$\Rightarrow \left\|\bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_n,t)\geq\beta\}\right\|\to 0$ as $n\to\infty$ for $p=1,2,3,\ldots$using (2)
$\Rightarrow \bigwedge\{t\succ\theta_E : N_c(x_{n+p}-x_n,t)\geq\beta\}\to\theta_E$ as $n\to\infty$ for $p=1,2,3,\ldots$
$\Rightarrow \{x_n\}$ is a β-Cauchy sequence.

$$So\ \exists x\in X \text{ such that } \lim_{n\to\infty}\bigwedge\{t\succ\theta_E : N_c(x_n-x,t)\geq\beta\}=\theta_E. \tag{3}$$

We have

$$\bigwedge\{s+t\succ\theta_E : N_c(Tx-x,s+t)\geq\beta*\beta\}$$
$$\preceq\bigwedge\{s\succ\theta_E : N_c(Tx-x_n,s)\geq\beta\}+\bigwedge\{t\succ\theta_E : N_c(x_n-x,t)\geq\beta\}$$
$$\preceq k\bigwedge\{s\succ\theta_E : N_c(x-x_{n-1},s)\geq\beta\}+\bigwedge\{t\succ\theta_E : N_c(x_n-x,t)\geq\beta\}$$

Using normality condition and (3), we get

$$\left\|\bigwedge\{s+t\succ\theta_E : N_c(Tx-x,s+t)\geq\beta*\beta\}\right\|$$
$$\leq M\left\|k\bigwedge\{s\succ\theta_E : N_c(x-x_{n-1},s)\geq\beta\}+\bigwedge\{t\succ\theta_E : N_c(x_n-x,t)\geq\beta\}\right\|$$
$$\leq Mk\left\|\bigwedge\{s\succ\theta_E : N_c(x-x_{n-1},s)\geq\beta\}\right\|+M\left\|\bigwedge\{t\succ\theta_E : N_c(x_n-x,t)\geq\beta\}\right\|$$

Taking limit as $n\to\infty$, we have

$$\left\|\bigwedge\{s+t\succ\theta_E : N_c(Tx-x,s+t)\geq\beta*\beta\}\right\|=0$$
$$\Rightarrow\bigwedge\{s+t\succ\theta_E : N_c(Tx-x,s+t)\geq\beta*\beta\}=\theta_E$$
$$\Rightarrow\forall(s+t)\succ\theta_E; N_c(Tx-x,s+t)\geq\beta*\beta>0$$
$$\Rightarrow Tx-x=\theta_X \text{ by } (FCN6)$$
$$\Rightarrow Tx=x.$$

Thus, x is a fixed point of T.
Uniqueness: If possible suppose that $\exists y\in X$ such that $Ty=y$.

$$\text{Now, }\bigwedge\{t\succ\theta_E : N_c(x-y,t)\geq\alpha\}=\bigwedge\{t\succ\theta_E : N_c(Tx-Ty,t)\geq\alpha\}$$
$$\preceq k\bigwedge\{t\succ\theta_E : N_c(x-y,t)\geq\alpha\}$$

Using normality condition with normal constant 1, we have

$$\left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| \leq k \left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\|$$
$$\Rightarrow (1-k)\left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| \leq 0$$
$$\Rightarrow \left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| = 0(\text{Since} 0 < k < 1)$$
$$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} = \theta_E$$
$$\Rightarrow \forall t \succ \theta_E; N_c(x-y,t) > 0$$
$$\Rightarrow x-y = \theta_X \text{ by } (FCN6)$$
$$\Rightarrow x = y.$$

Example 2 *Let us consider the l-fuzzy complete cone normed linear space* $(X, N_c, *)$ *of Example 1. Let T be self-map of X given by* $Tx = \frac{x}{3}$. *Take* $k = \frac{1}{2}$.

$$\text{Now, } \{t \succ \theta_E : N_c(Tx-Ty,t) \geq \alpha\}, \alpha \in (0,1)$$
$$= \left\{t \succ \theta_E : N_c\left(\frac{x}{3}-\frac{y}{3},t\right) \geq \alpha\right\}$$
$$= \{t \succ \theta_E : \frac{\|x-y\|_c}{3} \prec t\}$$
$$= \{t \succ \theta_E : \|x-y\|_c \prec 3t\}$$
$$\text{Again, } \left\{t \succ \theta_E : N_c\left(x-y,\frac{t}{k}\right) \geq \alpha\right\}$$
$$= \{t \succ \theta_E : N_c(x-y,2t) \geq \alpha\}$$
$$= \{t \succ \theta_E : \|x-y\|_c \prec 2t\}$$
$$\text{Thus, } \bigwedge\{t \succ \theta_E : \|x-y\|_c \prec 3t\} \preceq \bigwedge\{t \succ \theta_E : \|x-y\|_c \prec 2t\}$$
$$\text{i.e., } \bigwedge\{t \succ \theta_E : N_c(Tx-Ty,t) \geq \alpha\} \preceq \bigwedge\left\{t \succ \theta_E : N_c\left(x-y,\frac{t}{k}\right) \geq \alpha\right\}$$

Thus, T satisfies Banach type contraction. We see that 0 is the unique fixed point of T.

Theorem 5 (Kannan contraction type fixed point theorem in fuzzy cone normed linear space) .

Let $(X, N_c, *)$ be a l-fuzzy complete cone normed linear space where $* = \min$, P be a strongly minihedral cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition

$$\bigwedge\{t \succ \theta_E : N_c(Tx-Ty,t) \geq \alpha\} \preceq k\Big[\bigwedge\{t \succ \theta_E : N_c(Tx-x,t) \geq \alpha\}$$
$$+\bigwedge\{t \succ \theta_E : N_c(Ty-y,t) \geq \alpha\}\Big]$$

$\forall \alpha \in (0,1)$ and $k \in \left(0, \frac{1}{2}\right)$ is a constant. Then, T has a unique fixed point in X.

Proof Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$. So $\{x_n\}$ is a sequence in X. First, we show that $\{x_n\}$ is α-Cauchy sequence for all $\alpha \in (0,1)$. □

Now, for $\alpha \in$,

$$\begin{aligned}
&\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \\
&= \bigwedge\{t \succ \theta_E : N_c(Tx_n - Tx_{n-1}, t) \geq \alpha\} \\
&\preceq k\Big[\ \bigwedge\{t \succ \theta_E : N_c(Tx_n - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_{n-1}, t) \geq \alpha\}\Big] \\
&= k\Big[\ \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\}\Big] \\
&\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \frac{k}{1-k} \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} \\
&\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \delta^n \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\} \\
&\quad \text{where } \delta = \frac{k}{1-k},\ \ 0 < \delta < 1.
\end{aligned}$$

Using normality condition, we get,

$$\begin{aligned}
&\left\| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \right\| \leq M\delta^n \left\| \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\} \right\| \\
&\Rightarrow \lim_{n\to\infty} \left\| \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \right\| = 0. \\
&\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \to \theta_E \text{ as } n \to \infty
\end{aligned} \tag{4}$$

Now for $p \geq 1$ we have,

$$\begin{aligned}
&\bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+p} - x_{n+p-1}, \frac{t}{p}\right) \geq \alpha\right\} \\
&+ \bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+p-1} - x_{n+p-2}, \frac{t}{p}\right) \geq \alpha\right\} \\
&+ \ldots + \bigwedge\left\{t \succ \theta_E : N_c\left(x_{n+1} - x_n, \frac{t}{p}\right) \geq \alpha\right\} \\
&\succeq \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha * \alpha * \ldots * \alpha = \alpha\}
\end{aligned}$$

Using normality condition and (4), it follows that

$$\begin{aligned}
&\left\| \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \right\| \to 0 \text{ as } n \to \infty \text{ for } p = 1, 2, 3, \ldots \\
&\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+p} - x_n, t) \geq \alpha\} \to \theta_E \text{ as } n \to \infty \text{for } p = 1, 2, 3, \ldots \\
&\Rightarrow \{x_n\} \text{ is a } \alpha - \text{Cauchy sequence.}
\end{aligned}$$

Since X is l-fuzzy complete, thus $\exists x \in X$ such that

$$\lim_{n\to\infty} \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} = \theta_E \tag{5}$$

Now,

$$\bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha * \alpha\}$$
$$\preceq \bigwedge\{t \succ \theta_E : N_c(Tx - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\}$$
$$= \bigwedge\{t \succ \theta_E : N_c(Tx - Tx_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\}$$

i.e., $\bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\}$

$$\preceq k\Big[\bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\}\Big]$$
$$+ \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\}$$
$$\Rightarrow (1-k) \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\}$$
$$\preceq k \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\}$$

Using normality condition, from (4) and (5), we get

$$(1-k) \left\| \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} \right\|$$
$$\leq M \left\| k \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} \right\|$$
$$\leq Mk \left\| \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\} \right\| + M \left\| \bigwedge\{t \succ \theta_E : N_c(x_n - x, t) \geq \alpha\} \right\|$$
$$\Rightarrow \left\| \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} \right\| = 0 \text{ as } n \to \infty$$
$$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} = \theta_E$$
$$\Rightarrow N_c(Tx - x, t) \geq \alpha \;\forall\, \alpha \in (0,1) \text{ and } \forall\, t \succ \theta_E$$
$$\Rightarrow N_c(Tx - x, t) = 1 \;\forall\, t \succ \theta_E$$
$$\Rightarrow Tx - x = \theta_X \text{ by } (FCN2)$$
$$\Rightarrow Tx = x.$$

Thus, T has a fixed point.

Uniqueness: If $\exists\, y \in X$ such that $Ty = y$.

Now,

$$\bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\}$$
$$= \bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\}$$
$$\preceq k\Big[\bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Ty - y, t) \geq \alpha\}\Big]$$

Using normality condition, we have

$$\left\| \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} \right\|$$
$$\leq Mk \left\| \bigwedge\{t \succ \theta_E : N_c(Tx - x, t) \geq \alpha\} \right\| + Mk \left\| \bigwedge\{t \succ \theta_E : N_c(Ty - y, t) \geq \alpha\} \right\|$$
$$\Rightarrow \left\| \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} \right\| = 0.$$
$$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x - y, t) \geq \alpha\} = \theta_E.$$
$$\Rightarrow N_c(x - y, t) \geq \alpha \;\forall\, \alpha \in (0,1) \text{ and } \forall\, t \succ \theta_E$$
$$\Rightarrow N_c(x - y, t) = 1 \;\forall\, t \succ \theta_E$$
$$\Rightarrow x - y = \theta_X \text{ by } (FCN2)$$
$$\Rightarrow x = y.$$

Example 3 *Let us consider the l-fuzzy complete cone normed linear space* $(X, N_c, *)$ *of Example 1. Let T be self-map of X given by* $Tx = \frac{x}{6}$. *Take* $k = \frac{1}{5}$.

Now,

$$\begin{aligned}
&\bigwedge\left\{t \succ \theta_E : N_c\left(Tx - x, \frac{t}{k}\right) \geq \alpha\right\} + \bigwedge\left\{t \succ \theta_E : N_c\left(Ty - y, \frac{t}{k}\right) \geq \alpha\right\} \\
&\succeq \bigwedge\left\{s + t \succ \theta_E : N_c\left(Tx - x + y - Ty, \frac{s+t}{k}\right) \geq \alpha\right\} \\
&= \bigwedge\left\{s + t \succ \theta_E : N_c\left(\frac{5y}{6} - \frac{5x}{6}, \frac{s+t}{\frac{1}{5}}\right) \geq \alpha\right\} \\
&= \bigwedge\left\{s + t \succ \theta_E : N_c\left(\frac{x}{6} - \frac{y}{6}, s+t\right) \geq \alpha\right\} \\
&= \bigwedge\left\{t \succ \theta_E : N_c\left(\frac{x}{6} - \frac{y}{6}, t\right) \geq \alpha\right\} \\
&= \bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\}
\end{aligned}$$

Thus, T satisfies Kannan type contraction. We see that 0 is the unique fixed point of T.

Theorem 6 (Chatterjee contraction type fixed point theorem.)

Let $(X, N_c, *)$ be a l-fuzzy complete cone normed linear space where $* = \min$, P be a stronghly minihedral cone with normal constant M. Suppose the mapping $T : X \longrightarrow X$ satisfies the contractive condition

$$\begin{aligned}
&\bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\} \\
&\quad \preceq k\Big[\bigwedge\{t \succ \theta_E : N_c(Tx - y, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Ty - x, t) \geq \alpha\}\Big]
\end{aligned}$$

$\forall \alpha \in (0, 1)$ and $k \in (0, \frac{1}{2})$ is a constant. Then T has a fixed point in X. In addition, if $M = 1$, then the fixed point is unique.

Proof Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0$. So $\{x_n\}$ is a sequence in X. First, we show that $\{x_n\}$ is α-Cauchy sequence for all $\alpha \in (0, 1)$. □

Now for $\alpha \in (0, 1)$;
$\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\}$
$= \bigwedge\{t \succ \theta_E : N_c(Tx_n - Tx_{n-1}, t) \geq \alpha\}$
$\preceq k\big[\bigwedge\{t \succ \theta_E : N_c(Tx_n - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Tx_{n-1} - x_n, t) \geq \alpha\}\big]$
$= k\big[\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_{n-1}, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_n, t) \geq \alpha\}\big]$
$\preceq k\big[\bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\}\big] + \theta_E$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \frac{k}{1-k} \bigwedge\{t \succ \theta_E : N_c(x_n - x_{n-1}, t) \geq \alpha\}$
$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x_{n+1} - x_n, t) \geq \alpha\} \preceq \delta^n \bigwedge\{t \succ \theta_E : N_c(x_1 - x_0, t) \geq \alpha\}$ where $\delta = \frac{k}{1-k}, \; 0 < \delta < 1$.

Using normality condition, we get

$$\left\|\bigwedge\{t\succ\theta_E:N_c(x_{n+1}-x_n,t)\geq\alpha\}\right\|\leq M\delta^n\left\|\bigwedge\{t\succ\theta_E:N_c(x_1-x_0,t)\geq\alpha\}\right\|$$
$$\Rightarrow\lim_{n\to\infty}\left\|\bigwedge\{t\succ\theta_E:N_c(x_{n+1}-x_n,t)\geq\alpha\}\right\|=0.$$
$$\Rightarrow\lim_{n\to\infty}\bigwedge\{t\succ\theta_E:N_c(x_{n+1}-x_n,t)\geq\alpha\}=\theta_E \tag{6}$$

Now for $p\geq 1$, we have

$$\bigwedge\{t\succ\theta_E:N_c\left(x_{n+p}-x_{n+p-1},\frac{t}{p}\right)\geq\alpha\}+\bigwedge\left\{t\succ\theta_E:N_c\left(x_{n+p-1}-x_{n+p-2},\frac{t}{p}\right)\geq\alpha\right\}$$
$$+\ldots+\bigwedge\left\{t\succ\theta_E:N_c\left(x_{n+1}-x_n,\frac{t}{p}\right)\geq\alpha\right\}\succeq\bigwedge\{t\succ\theta_E:N_c(x_{n+p}-x_n,t)\geq\alpha*\alpha*\ldots*\alpha=\alpha\} \tag{7}$$

Using normality condition and (6), it follows that

$$\left\|\bigwedge\{t\succ\theta_E:N_c(x_{n+p}-x_n,t)\geq\alpha\}\right\|\to 0\text{ as }n\to\infty\text{ for }p=1,2,3,\ldots$$
$$\Rightarrow\bigwedge\{t\succ\theta_E:N_c(x_{n+p}-x_n,t)\geq\alpha\}\to\theta_E\text{ as }n\to\infty\text{ for }p=1,2,3,\ldots$$
$$\Rightarrow\{x_n\}\text{ is a}\alpha-\text{Cauchy sequence.}$$

Since X is l-fuzzy complete, thus $\exists x\in X$ such that

$$\lim_{n\to\infty}\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}=\theta_E \tag{8}$$

Now,

$$\bigwedge\{t\succ\theta_E:N_c(Tx-x,t)\geq\alpha*\alpha\}$$
$$\preceq\bigwedge\{t\succ\theta_E:N_c(Tx-x_n,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$
$$=\bigwedge\{t\succ\theta_E:N_c(Tx-Tx_{n-1},t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$
$$\preceq k\Big[\bigwedge\{t\succ\theta_E:N_c(Tx-x_{n-1},t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(Tx_{n-1}-x,t)\geq\alpha\}\Big]$$
$$+\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$
$$=k\bigwedge\{t\succ\theta_E:N_c(Tx-x_{n-1},t)\geq\alpha\}+(1+k)\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$
$$\preceq k\Big[\bigwedge\{t\succ\theta_E:N_c(Tx-x,t)\geq\alpha\}+\bigwedge\{t\succ\theta_E:N_c(x_{n-1}-x,t)\geq\alpha\}\Big]$$
$$+(1+k)\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$
$$\Rightarrow(1-k)\bigwedge\{t\succ\theta_E:N_c(Tx-x,t)\geq\alpha\}\preceq k\bigwedge\{t\succ\theta_E:N_c(x_{n-1}-x,t)\geq\alpha\}$$
$$+(1+k)\bigwedge\{t\succ\theta_E:N_c(x_n-x,t)\geq\alpha\}$$

Using normality condition and (8), we get

$$\left\|(1-k)\bigwedge\{t \succ \theta_E : N_c(Tx-x,t) \geq \alpha\}\right\|$$
$$\leq M\left\|k\bigwedge\{t \succ \theta_E : N_c(x_{n-1}-x,t) \geq \alpha\}\right] + (1+k)\bigwedge\{t \succ \theta_E : N_c(x_n-x,t) \geq \alpha\}\right\|$$
$$\Rightarrow \left\|\bigwedge\{t \succ \theta_E : N_c(Tx-x,t) \geq \alpha\}\right\| \leq \frac{Mk}{1-k}\left\|\bigwedge\{t \succ \theta_E : N_c(x_{n-1}-x,t) \geq \alpha\}\right\|$$
$$+\frac{M(1+k)}{1-k}\left\|\bigwedge\{t \succ \theta_E : N_c(x_n-x,t) \geq \alpha\}\right\|$$
$$\Rightarrow \left\|\bigwedge\{t \succ \theta_E : N_c(Tx-x,t) \geq \alpha\}\right\| = 0 \text{ as } n \to \infty$$
$$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(Tx-x,t) \geq \alpha\} = \theta_E$$
$$\Rightarrow N_c(Tx-x,t) \geq \alpha \ \forall\, \alpha \in (0,1) \text{ and } \forall\, t \succ \theta_E$$
$$\Rightarrow N_c(Tx-x,t) = 1 \ \forall\, t \succ \theta_E$$
$$\Rightarrow Tx - x = \theta_X \text{ by } (FCN2)$$
$$\Rightarrow Tx = x.$$

Thus, T has a fixed point.
Uniqueness: If $\exists\, y \in X$ such that $Ty = y$.
Now,

$$\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}$$
$$= \bigwedge\{t \succ \theta_E : N_c(Tx-Ty,t) \geq \alpha\}$$
$$\preceq k\left[\bigwedge\{t \succ \theta_E : N_c(Tx-y,t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(Ty-x,t) \geq \alpha\}\right]$$
$$= k\left[\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} + \bigwedge\{t \succ \theta_E : N_c(y-x,t) \geq \alpha\}\right]$$
$$= 2k\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}$$

Using normality condition with normal constant 1, we get

$$\left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| \leq \left\|2k\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\|$$
$$\Rightarrow (1-2k)\left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| \leq 0$$
$$\Rightarrow \left\|\bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\}\right\| = 0. \left(\text{Since } 0 < k < \frac{1}{2}\right)$$
$$\Rightarrow \bigwedge\{t \succ \theta_E : N_c(x-y,t) \geq \alpha\} = \theta_E.$$
$$\Rightarrow N_c(x-y,t) \geq \alpha \ \forall\, \alpha \in (0,1) \text{ and } \forall\, t \succ \theta_E$$
$$\Rightarrow N_c(x-y,t) = 1 \ \forall\, t \succ \theta_E$$
$$\Rightarrow x - y = \theta_X \text{ by } (FCN2)$$
$$\Rightarrow x = y.$$

Example 4 *Let us consider the l-fuzzy complete cone normed linear space* $(X, N_c, *)$ *of Example 1. Let T be self-map of X given by* $Tx = \frac{x}{2}$. *Take* $k = \frac{1}{3}$.

Now,

$$\bigwedge\left\{t \succ \theta_E : N_c\left(Tx - y, \frac{t}{k}\right) \geq \alpha\right\} + \bigwedge\left\{t \succ \theta_E : N_c\left(Ty - x, \frac{t}{k}\right) \geq \alpha\right\}$$
$$\geq \bigwedge\left\{s + t \succ \theta_E : N_c\left(Tx - y + x - Ty, \frac{s+t}{k}\right) \geq \alpha\right\}$$
$$= \bigwedge\left\{s + t \succ \theta_E : N_c\left(\frac{3x}{2} - \frac{3y}{2}, \frac{s+t}{\frac{1}{3}}\right) \geq \alpha\right\}$$
$$= \bigwedge\left\{s + t \succ \theta_E : N_c\left(\frac{x}{2} - \frac{y}{2}, s+t\right) \geq \alpha\right\}$$
$$= \bigwedge\left\{t \succ \theta_E : N_c\left(\frac{x}{2} - \frac{y}{2}, t\right) \geq \alpha\right\}$$
$$= \bigwedge\{t \succ \theta_E : N_c(Tx - Ty, t) \geq \alpha\}$$

Thus, T satisfies Chatterjee type contraction. We see that 0 is the unique fixed point of T.

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