El-Deeb and Bulboacă *Journal of the Egyptian Mathematical Society* (2019) 27:42
https://doi.org/10.1186/s42787-019-0049-2

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH Open Access

Fekete-Szegő inequalities for certain class of analytic functions connected with *q*-analogue of Bessel function

Sheza M. El-Deeb[1,2*] and Teodor Bulboacă[3]

*Correspondence:
shezaeldeeb@yahoo.com
[1]Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
[2]Department of Mathematics, Faculty of Science and Arts in Badaya, Al-Qassim University, Al-Badaya, Al-Qassim, Kingdom of Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we obtain Fekete-Szegő inequalities for a certain class of analytic functions f satisfying $1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}f(z)+\gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}-1\right]\prec\Psi(z)$. Application of our results to certain functions defined by convolution products with a normalized analytic function is given, and in particular, Fekete-Szegő inequalities for certain subclasses of functions defined through Poisson distribution are obtained.

Keywords: Fekete-Szegő inequality, Differential subordination, Bessel function of first kind, q-derivative, Poisson distribution

2010 Mathematics Subject Classification: 30C45, 30C80

Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k,\ z\in\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\},\tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent functions in $\mathbb{D}$.

If $k\in\mathcal{A}$ is given by:

$$k(z)=z+\sum_{k=2}^{\infty}b_kz^k,\ z\in\mathbb{D},\tag{2}$$

then, the *Hadamard (or convolution) product* of f and k is defined by:

$$(f\times k)(z):=z+\sum_{k=2}^{\infty}a_kb_kz^k,\ z\in\mathbb{D}.\tag{3}$$

If f and F are analytic functions in $\mathbb{D}$, we say that f *is subordinate to* F, written $f\prec F$, if there exists a *Schwarz function* w, which is analytic in $\mathbb{D}$, with $w(0)=0$, and $|w(z)|<1$ for all $z\in\mathbb{D}$, such that $f(z)=F(w(z))$, $z\in\mathbb{D}$. Furthermore, if the function F is univalent in $\mathbb{D}$, then we have the following equivalence (see [1] and [2]):

$$f(z)\prec F(z)\Leftrightarrow f(0)=F(0)\text{ and }f(\mathbb{D})\subset F(\mathbb{D}).$$

The *Bessel function of the first kind of order* ν is defined by the infinite series:

$$J_\nu(z) := \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)},\ z\in\mathbb{C},\quad (\nu\in\mathbb{R}),$$

where Γ stands for the *Gamma function*. Recently, Szász and Kupán [3] investigated the univalence of the normalized Bessel function of the first kind $g_\nu:\mathbb{D}\to\mathbb{C}$ defined by (see also [4–6])

$$g_\nu(z) := 2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}J_\nu(z^{\frac{1}{2}})$$
$$= z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\,\Gamma(k+\nu)}z^k,\ z\in\mathbb{D},\quad (\nu\in\mathbb{R}).$$

For $0<q<1$, the *q-derivative operator* for g_ν is defined by:

$$\partial_q g_\nu(z) = \partial_q\left[z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\,\Gamma(k+\nu)}z^k\right] := \frac{g_\nu(qz)-g_\nu(z)}{z(q-1)} =$$
$$1+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\,\Gamma(k+\nu)}[k,q]\,z^{k-1},\ z\in\mathbb{D},$$

where

$$[k,q] := \frac{1-q^k}{1-q} = 1+\sum_{j=1}^{k-1}q^j,\qquad [0,q]:=0. \tag{4}$$

Using definition formula (4), we will define the next two products:
(i) For any non-negative integer k, the *q-shifted factorial* is given by:

$$[k,q]! := \begin{cases} 1, & \text{if } k=0,\\ [1,q]\,[2,q]\,[3,q]\ldots[k,q], & \text{if } k\in\mathbb{N}.\end{cases}$$

(ii) For any positive number r, the *q-generalized Pochhammer symbol* is defined by:

$$[r,q]_k := \begin{cases} 1, & \text{if } k=0,\\ [r,q]\,[r+1,q]\ldots[r+k-1,q], & \text{if } k\in\mathbb{N}.\end{cases}$$

For $\nu>0$, $\lambda>-1$, and $0<q<1$, define the function $\mathcal{I}^\lambda_{\nu,q}:\mathbb{D}\to\mathbb{C}$ by:

$$\mathcal{I}^\lambda_{\nu,q}(z) := z+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\,\Gamma(k+\nu)}\,\frac{[k,q]!}{[\lambda+1,q]_{k-1}}z^k,\ z\in\mathbb{D}.$$

Remark 1 *A simple computation shows that:*

$$\mathcal{I}^\lambda_{\nu,q}(z)\times\mathcal{M}_{q,\lambda+1}(z) = z\,\partial_q g_\nu(z),\ z\in\mathbb{D},$$

where the function $\mathcal{M}_{q,\lambda+1}$ *is given by:*

$$\mathcal{M}_{q,\lambda+1}(z) := z+\sum_{k=2}^{\infty}\frac{[\lambda+1,q]_{k-1}}{[k-1,q]!}z^k,\ z\in\mathbb{D}.$$

Using the definition of q-derivative along with the idea of convolutions, we introduce the linear operator $\mathcal{N}_{\nu,q}^{\lambda}:\mathcal{A}\to\mathcal{A}$ defined by:

$$\mathcal{N}_{\nu,q}^{\lambda}f(z):=\mathcal{I}_{\nu,q}^{\lambda}(z)\times f(z)=z+\sum_{k=2}^{\infty}\psi_k a_k z^k,\ z\in\mathbb{D}, \tag{5}$$
$$(\nu>0,\ \lambda>-1,\ 0<q<1),$$

where

$$\psi_k:=\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\,\Gamma(k+\nu)}\cdot\frac{[k,q]\,!}{[\lambda+1,q]_{k-1}}. \tag{6}$$

Remark 2 *From definition relation* (5)*, we can easily verify that the next relations hold for all* $f\in\mathcal{A}$*:*
(i) $[\lambda+1,q]\,\mathcal{N}_{\nu,q}^{\lambda}f(z)=[\lambda,q]\,\mathcal{N}_{\nu,q}^{\lambda+1}f(z)+q^{\lambda}z\partial_q\left(\mathcal{N}_{\nu,q}^{\lambda+1}f(z)\right),z\in\mathbb{D}$*;*
(ii) $\lim\limits_{q\to1^-}\mathcal{N}_{\nu,q}^{\lambda}f(z)=\mathcal{I}_{\nu,1}^{\lambda}\times f(z)=:\mathcal{I}_{\nu}^{\lambda}f(z)=$
$z+\sum\limits_{k=2}^{\infty}\frac{k!}{(\lambda+1)_{k-1}}\frac{(-1)^{k-1}\Gamma(\nu+1)}{4^{k-1}(k-1)!\Gamma(k+\nu)}a_kz^k,z\in\mathbb{D}.$

Now, we define the class of functions $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$ as follows:

Definition 1 *Let* $\Psi(z):=1+B_1z+B_2z^2+\ldots,\ z\in\mathbb{D}$*, with* $B_1>0$*, be a starlike (univalent) function with respect to 1, which maps the unit disk* $\mathbb{D}$ *onto a region included in the right half plane which is symmetric with respect to the real axis. For* $\zeta\in\mathbb{C}^*$*, and* $0\le\gamma<1$*, the function* $f\in\mathcal{A}$ *is said to be in the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$ *if the function*

$$1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}f(z)+\gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}-1\right]$$

is analytic in $\mathbb{D}$ *and satisfies:*

$$1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}f(z)+\gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}-1\right]\prec\Psi(z)$$
$$\left(\nu>0,\ \lambda>-1,\ 0<q<1,\ \zeta\in\mathbb{C}^*,\ 0\le\gamma<1\right).$$

Putting $q\to1^-$, we obtain that $\lim\limits_{q\to1^-}\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)=:\mathcal{G}_{\nu}^{\lambda,\gamma}(\zeta;\Psi)$, where

$$\mathcal{G}_{\nu}^{\lambda,\gamma}(\zeta;\Psi):=\left\{1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{I}_{\nu}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{I}_{\nu}^{\lambda}f(z)+\gamma z\left(\mathcal{I}_{\nu}^{\lambda}f(z)\right)'}-1\right]\prec\Psi(z)\right\}$$
$$\left(\nu>0,\ \lambda>-1,\ \zeta\in\mathbb{C}^*,\ 0\le\gamma<1\right).$$

In this paper, we obtain the Fekete-Szegő inequalities for the functions of the class $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$. We give some application of our results to certain functions defined by convolution products with a normalized analytic function. In particular, Fekete-Szegő inequalities for certain subclasses of functions defined through Poisson distribution are obtained.

Fekete-Szegő problem

Denoted by $\mathcal{P}$, the well-known *Carathéodory's* class of analytic functions in $\mathbb{D}$, normalized with $P(0) = 1$, and having positive real part in $\mathbb{D}$, that is $\operatorname{Re}P(z) > 0$ for all $z \in \mathbb{D}$ (see [7]).

To prove our results, we need the following two lemmas.

Lemma 1 [8, Lemma 3] *If $p(z) = 1 + c_1z + c_2z^2 + \dots \in \mathcal{P}$, and α is a complex number, then*

$$\max\left|c_2 - \alpha c_1^2\right| = 2\max\{1; |2\alpha - 1|\}.$$

Lemma 2 [9, Lemma 1] *If $p(z) = 1 + c_1z + c_2z^2 + \dots \in \mathcal{P}$, then*

$$\left|c_2 - \alpha c_1^2\right| \le \begin{cases} -4\alpha + 2, & \text{if } \alpha \le 0, \\ 2, & \text{if } 0 \le \alpha \le 1, \\ 4\alpha - 2, & \text{if } \alpha \ge 1. \end{cases}$$

When $\alpha < 0$ or $\alpha > 1$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations.
If $0 < \alpha < 1$, then the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations.
If $\alpha = 0$, the equality holds if and only if:

$$p(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}, \quad \textit{with} \quad 0 \le \lambda \le 1,$$

or one of its rotations.
If $\alpha = 1$, the equality holds if and only if:

$$\frac{1}{p(z)} = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}, \quad \textit{with} \quad 0 \le \lambda \le 1.$$

Like it was mentioned in [9, pages 162–163], although the above upper bound is sharp, it can be improved as follows when $0 < \alpha < 1$:

$$\left|c_2 - \alpha c_1^2\right| + \alpha\left|c_1\right|^2 \le 2, \quad \text{if} \quad 0 < \alpha \le \frac{1}{2}, \tag{7}$$

and

$$\left|c_2 - \alpha c_1^2\right| + (1 - \alpha)\left|c_1\right|^2 \le 2, \quad \text{if} \quad \frac{1}{2} \le \alpha < 1. \tag{8}$$

Theorem 1 *If the function f given by (1) belongs to the class $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$, with $\Psi(z) = 1 + B_1z + B_2z^2 + \dots$ satisfying the conditions of Definition 1, and μ is a complex number, then:*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\zeta|B_1}{2(1-\gamma)\psi_3} \cdot \max\left\{1; \left|\frac{B_2}{B_1} + \frac{\zeta B_1(1+\gamma)}{1-\gamma} - \frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right|\right\},$$

where ψ_k, $k \in \{2, 3\}$, are given by (6).

Proof If $f \in \mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$, then there exists a Schwarz function w, that is w is analytic in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that:

$$1 + \frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}f(z) + \gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'} - 1\right] = \Psi(w(z)), \; z \in \mathbb{D}. \tag{9}$$

Since w is a Schwarz function, it follows that the function p_1 defined by:

$$p_1(z) := \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots,\ z \in \mathbb{D}, \tag{10}$$

belongs to $\mathcal{P}$. Defining the function p by:

$$p(z) := 1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}f(z)+\gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}f(z)\right)'}-1\right] = 1+d_1z+d_2z^2+\ldots,\ z \in \mathbb{D}, \tag{11}$$

in view of (9) and (10), we have:

$$p(z) = \Psi\left(\frac{p_1(z)-1}{p_1(z)+1}\right),\ z \in \mathbb{D}. \tag{12}$$

From (10), we easily get:

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 + \frac{c_1^3}{4} - c_1c_2\right)z^3 + \ldots\right],\ z \in \mathbb{D};$$

therefore,

$$\Psi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right]z^2 + \ldots,\ z \in \mathbb{D},$$

and from (12), we obtain:

$$d_1 = \frac{1}{2}B_1c_1 \qquad \text{and} \qquad d_2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2. \tag{13}$$

On the other hand, from (11), according to (5), it follows that

$$d_1 = \frac{(1-\gamma)a_2\psi_2}{\zeta} \qquad \text{and} \qquad d_2 = \frac{2(1-\gamma)a_3\psi_3}{\zeta} - \frac{(1-\gamma)(1+\gamma)a_2^2\psi_2^2}{\zeta}, \tag{14}$$

and combining (13) with (14), we have:

$$a_2 = \frac{\zeta B_1 c_1}{2(1-\gamma)\psi_2}, \tag{15}$$

and

$$a_3 = \frac{\zeta B_1}{4(1-\gamma)\psi_3}\left[c_2 - \frac{c_1^2}{2} + \frac{1}{2}\frac{B_2}{B_1}c_1^2 + \frac{\zeta B_1(1+\gamma)c_1^2}{2(1-\gamma)}\right].$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{\zeta B_1}{4(1-\gamma)\psi_3}\left(c_2 - \alpha c_1^2\right), \tag{16}$$

where

$$\alpha = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{\zeta B_1(1+\gamma)}{1-\gamma} + \frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right], \tag{17}$$

and from Lemma 1, our result follows immediately. □

Putting $q \to 1^-$ in Theorem 1, we obtain the next corollary:

Corollary 1 *If the function f given by (1) belongs to the class $\mathcal{G}_{\nu}^{\lambda,\gamma}(\zeta;\Psi)$, with $\Psi(z) = 1 + B_1z + B_2z^2 + \ldots$ satisfying the conditions of Definition 1, and μ is a complex number, then*

$$\left|a_3-\mu a_2^2\right| \leq \frac{8|\zeta|B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)}\cdot\max\left\{1;\left|\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{3\mu\zeta B_1(\lambda+1)(\nu+1)}{2(1-\gamma)(\lambda+2)(\nu+2)}\right|\right\}.$$

Using a similar proof like for Theorem 1 combined with Lemma 2, we can obtain the following theorem:

Theorem 2 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$ *satisfying the conditions of Definition* 1 *and* $\mu, B_2\in\mathbb{R}$, *and* $\zeta>0$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\zeta B_1}{2(1-\gamma)\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right], & \text{if } \mu\leq\sigma_1,\\ \frac{\zeta B_1}{2(1-\gamma)\psi_3}, & \text{if } \sigma_1\leq\mu\leq\sigma_2,\\ \frac{-\zeta B_1}{2(1-\gamma)\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right], & \text{if } \mu\geq\sigma_2,\end{cases}$$

with

$$\sigma_1=\frac{(1-\gamma)\psi_2^2}{2\zeta B_1\psi_3}\left[-1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right], \tag{18}$$

and

$$\sigma_2=\frac{(1-\gamma)\psi_2^2}{2\zeta B_1\psi_3}\left[1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right], \tag{19}$$

where ψ_k, $k\in\{2,3\}$, *are given by* (6).

Proof With the same proof like those of Theorem 1, we obtain the equalities (16) and (17) hold.

(i) According to the first part of Lemma 2, we have:

$$\left|c_2-\alpha c_1^2\right|\leq -4\alpha+2,\ \text{if } \alpha\leq 0.$$

Using (17), simple computation shows that the inequality $\alpha\leq 0$ is equivalent to $\mu\leq\sigma_1$, and from (16) combined with the inequality $\left|c_2-\alpha c_1^2\right|\leq -4\alpha+2$, the first of our theorem is proved.

(ii) The second part of Lemma 2 shows that:

$$\left|c_2-\alpha c_1^2\right|\leq 2,\ \text{if } 0\leq\alpha\leq 1.$$

From (17), it is easy to check that the inequality $0\leq\alpha\leq 1$ is equivalent to $\sigma_1\leq\mu\leq\sigma_2$. From the relation (16), the inequality $\left|c_2-\alpha c_1^2\right|\leq 2$ proves the second part of our result.

(iii) Finally, form the third part of Lemma 2, we have:

$$\left|c_2-\alpha c_1^2\right|\leq 4\alpha-2,\ \text{if } \alpha\geq 1.$$

The relation (17) shows immediately that $\alpha\geq 1$ is equivalent to $\mu\geq\sigma_2$, while (16) combined with the inequality $\left|c_2-\alpha c_1^2\right|\leq 4\alpha-2$ proves the last part of our result. □

Taking $q\to 1^-$ in Theorem 2, we get the next special case:

Corollary 2 *If the function f given by* (1) *belongs to the class* $\mathcal{G}_{\nu}^{\lambda,\gamma}(\zeta;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$ *satisfying the conditions of Definition* 1 *and* $\mu, B_2\in\mathbb{R}$, *and* $\zeta>0$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{8\zeta B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{3\mu\zeta B_1(\lambda+1)(\nu+1)}{2(1-\gamma)(\lambda+2)(\nu+2)}\right], & \\ \hfill \textit{if} \quad \mu \leq \eta_1, & \\ \frac{8\zeta B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)}, \hfill \textit{if} \quad \eta_1 \leq \mu \leq \eta_2, & \\ \frac{-8\zeta B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{3\mu\zeta B_1(\lambda+1)(\nu+1)}{2(1-\gamma)(\lambda+2)(\nu+2)}\right], & \\ \hfill \textit{if} \quad \mu \geq \eta_2, & \end{cases}$$

with

$$\eta_1=\frac{2(1-\gamma)(\lambda+2)(\nu+2)}{3\zeta B_1(\lambda+1)(\nu+1)}\left[-1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}\right], \tag{20}$$

and

$$\eta_2=\frac{2(1-\gamma)(\lambda+2)(\nu+2)}{3\zeta B_1(\lambda+1)(\nu+1)}\left[1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}\right]. \tag{21}$$

With a similar proof like for Theorem 1 and using the inequalities (7) and (8), we obtained the next result.

Theorem 3 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$ *satisfying the conditions of Definition* 1 *and* $\mu, B_2 \in \mathbb{R}$, *and* $\zeta>0$, *then the next inequalities hold:*

(i) for $\sigma_1<\mu\leq\sigma_3$, *we have*

$$\left|a_3-\mu a_2^2\right|+\frac{(1-\gamma)\psi_2^2}{2\zeta B_1\psi_3}\left[1-\frac{B_2}{B_1}-\frac{\zeta B_1(1+\gamma)}{1-\gamma}+\frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right]|a_2|^2 \leq \frac{\zeta B_1}{2(1-\gamma)\psi_3}; \tag{22}$$

(ii) for $\sigma_3\leq\mu\leq\sigma_2$, *we have*

$$\left|a_3-\mu a_2^2\right|+\frac{(1-\gamma)\psi_2^2}{2\zeta B_1\psi_3}\left[1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{2\mu\zeta B_1\psi_3}{(1-\gamma)\psi_2^2}\right]|a_2|^2 \leq \frac{\zeta B_1}{2(1-\gamma)\psi_3}, \tag{23}$$

where σ_1 *and* σ_2 *are defined by* (18) *and* (19), *respectively,*

$$\sigma_3=\frac{(1-\gamma)\psi_2^2}{2\zeta B_1\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}\right],$$

and ψ_k, $k\in\{2,3\}$, *are given by* (6).

Proof With the same computations like in the proof of Theorem 1, we obtain the relations (16) and (17), while (15) is equivalent to:

$$c_1=\frac{2(1-\gamma)\psi_2}{\zeta B_1}. \tag{24}$$

(i) To prove the first part of our theorem, we will use the inequality (7). Thus, according to (16), (17), and the above relation, it is easy to check that (7) could be written in the equivalent form (22), while the assumption $0<\alpha\leq\frac{1}{2}$ is equivalent to $\sigma_1<\mu\leq\sigma_3$.

(ii) For the proof of the second part of our result, we will use the inequality (8). From (16), (17), and (24), it follows that (8) could be written in the form (23), and the assumption $\frac{1}{2}\leq\alpha<1$ is equivalent to $\sigma_3<\mu\leq\sigma_2$. □

Putting $q \to 1^-$ in Theorem 3, we obtain the following result:

Corollary 3 *If the function f given by* (1) *belongs to the class* $\mathcal{G}_{\nu}^{\lambda,\gamma}(\zeta;\Psi)$, *with* $\Psi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ *satisfying the conditions of Definition* 1 *and* $\mu, B_2 \in \mathbb{R}$, *and* $\zeta > 0$, *then the next inequalities hold:*
(i) for $\eta_1 < \mu \leq \eta_3$, *we have*

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| \\ &+\frac{2(1-\gamma)(\lambda+2)(\nu+2)}{3\zeta B_1(\lambda+1)(\nu+1)}\left[1 - \frac{B_2}{B_1} - \frac{\zeta B_1(1+\gamma)}{1-\gamma} + \frac{3\mu\zeta B_1(\lambda+1)(\nu+1)}{2(1-\gamma)(\lambda+2)(\nu+2)}\right]|a_2|^2 \\ &\leq \frac{8\zeta B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)};\end{aligned}$$

(ii) for $\eta_3 \leq \mu \leq \eta_2$, *we have*

$$\begin{aligned}&\left|a_3 - \mu a_2^2\right| \\ &+\frac{2(1-\gamma)(\lambda+2)(\nu+2)}{3\zeta B_1(\lambda+1)(\nu+1)}\left[1 + \frac{B_2}{B_1} + \frac{\zeta B_1(1+\gamma)}{1-\gamma} - \frac{3\mu\zeta B_1(\lambda+1)(\nu+1)}{2(1-\gamma)(\lambda+2)(\nu+2)}\right]|a_2|^2 \\ &\leq \frac{8\zeta B_1(\lambda+1)_2(\nu+1)_2}{3(1-\gamma)},\end{aligned}$$

where η_1 *and* η_2 *are defined by* (20) *and* (21), *respectively, and*

$$\eta_3 = \frac{2(1-\gamma)(\lambda+2)(\nu+2)}{3\zeta B_1(\lambda+1)(\nu+1)}\left[\frac{B_2}{B_1} + \frac{\zeta B_1(1+\gamma)}{(1-\gamma)}\right].$$

Applications to functions defined by poisson distribution

In [10], Porwal studied a power series whose coefficients are probabilities of the Poisson distribution, that is:

$$\mathrm{I}_m(z) = z + \sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}z^k,\ z \in \mathbb{D}, \quad (m > 0),$$

and motivated by this investigation Srivastava and Porwal [11] introduced the linear operator $\mathcal{I}^m : \mathcal{A} \to \mathcal{A}$ defined by:

$$\mathcal{I}^m f(z) := \mathrm{I}_m(z) \times f(z) = z + \sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}e^{-m}a_k z^k,\ z \in \mathbb{D},$$

where $f \in \mathcal{A}$ has the form (1).

Definition 2 *Let the function* Ψ *satisfying the conditions of Definition* 1. *For* $\zeta \in \mathbb{C}^*$, $0 \leq \gamma < 1$, *and* $k \in \mathcal{A}$, *the function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;k;\Psi)$ *if* $f \times k \in \mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$, *that is"*

$$1 + \frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}(f \times k)(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}(f \times k)(z) + \gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}(f \times k)(z)\right)'} - 1\right]$$

is analytic in $\mathbb{D}$ *and satisfies*

$$1+\frac{1}{\zeta}\left[\frac{z\left(\mathcal{N}_{\nu,q}^{\lambda}(f\times k)(z)\right)'}{(1-\gamma)\mathcal{N}_{\nu,q}^{\lambda}(f\times k)(z)+\gamma z\left(\mathcal{N}_{\nu,q}^{\lambda}(f\times k)(z)\right)'}-1\right]\prec\Psi(z)$$
$$\left(\nu>0,\ \lambda>-1,\ 0<q<1,\ \zeta\in\mathbb{C}^*,\ 0\le\gamma<1\right).$$

A special case of the class $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;k;\Psi)$ is obtained for $k=\mathrm{I}_m$; hence, $f\in\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\mathrm{I}_m;\Psi)$ if and only if $\mathcal{I}^m f\in\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\Psi)$.

Applying Theorems 1 and 2 for the function $f\times k$ given by (3), we get the following results, respectively:

Theorem 4 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;k;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$, $k\in\mathcal{A}$ *is given by* (2) *with* $b_2b_3\neq0$, *and* μ *is a complex number, then*

$$\left|a_3-\mu a_2^2\right|\le\frac{|\zeta|B_1}{2(1-\gamma)|b_3|\psi_3}\cdot\max\left\{1,\left|\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{2\mu\zeta B_1b_3\psi_3}{(1-\gamma)b_2^2\psi_2^2}\right|\right\},$$

where ψ_k *and* $k\in\{2,3\}$ *are given by* (6).

Theorem 5 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;k;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$ *satisfying the conditions of Definition* 1 *and* $\mu,B_2\in\mathbb{R}$, $k\in\mathcal{A}$ *is given by* (2) *with* $b_2b_3\neq0$, *and* $\zeta>0$, *then*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\frac{\zeta B_1}{2(1-\gamma)|b_3|\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}-\frac{2\mu\zeta B_1b_3\psi_3}{(1-\gamma)b_2^2\psi_2^2}\right], & \text{if } \mu\le\sigma_1,\\ \frac{\zeta B_1}{2(1-\gamma)|b_3|\psi_3}, & \text{if } \sigma_1\le\mu\le\sigma_2,\\ \frac{-\zeta B_1}{2(1-\gamma)|b_3|\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}-\frac{2\mu\zeta B_1b_3\psi_3}{(1-\gamma)b_2^2\psi_2^2}\right], & \text{if } \mu\ge\sigma_2,\end{cases}$$

with

$$\sigma_1=\frac{(1-\gamma)b_2^2\psi_2^2}{2\zeta B_1b_3\psi_3}\left[-1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right],$$

and

$$\sigma_2=\frac{(1-\gamma)b_2^2\psi_2^2}{2\zeta B_1b_3\psi_3}\left[1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right],$$

and ψ_k, $k\in\{2,3\}$, *are given by* (6).

For $k:=\mathrm{I}_m$, we have

$$b_2=me^{-m}\quad\text{and}\quad b_3=\frac{m^2}{2}e^{-m},$$

and for this special case from Theorems 4 and 5, we deduce to the following results, respectively:

Theorem 6 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\mathrm{I}_m;\Psi)$, *with* $\Psi(z)=1+B_1z+B_2z^2+\ldots$, *and* μ *is a complex number, then*

$$\left|a_3-\mu a_2^2\right|\le\frac{|\zeta|B_1}{(1-\gamma)m^2e^{-m}\psi_3}\cdot\max\left\{1;\left|\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}-\frac{\mu\zeta B_1\psi_3}{(1-\gamma)e^{-m}\psi_2^2}\right|\right\},$$

where ψ_k *and* $k \in \{2,3\}$ *are given by* (6).

Theorem 7 *If the function f given by* (1) *belongs to the class* $\mathcal{M}_{\nu,q}^{\lambda,\gamma}(\zeta;\mathrm{I}_m;\Psi)$, *with* $\Psi(z) = 1+B_1z+B_2z^2+\ldots$ *satisfying the conditions of Definition* 1 *and* $\mu, B_2 \in \mathbb{R}$, *and* $\zeta > 0$, *then*

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{\zeta B_1}{(1-\gamma)m^2e^{-m}\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}-\frac{\mu\zeta B_1\psi_3}{(1-\gamma)e^{-m}\psi_2^2}\right], & \text{if} \quad \mu \leq \sigma_1^*, \\ \frac{\zeta B_1}{(1-\gamma)m^2e^{-m}\psi_3}, & \text{if} \quad \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \frac{-\zeta B_1}{(1-\gamma)m^2e^{-m}\psi_3}\left[\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{(1-\gamma)}-\frac{\mu\zeta B_1\psi_3}{(1-\gamma)e^{-m}\psi_2^2}\right], & \text{if} \quad \mu \geq \sigma_2^*, \end{cases}$$

with

$$\sigma_1^* = \frac{(1-\gamma)e^{-m}\psi_2^2}{\zeta B_1\psi_3}\left[-1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right],$$

and

$$\sigma_2^* = \frac{(1-\gamma)e^{-m}\psi_2^2}{\zeta B_1\psi_3}\left[1+\frac{B_2}{B_1}+\frac{\zeta B_1(1+\gamma)}{1-\gamma}\right],$$

where ψ_k *and* $k \in \{2,3\}$ *are given by* (6).

Acknowledgements
The authors are grateful to the reviewer of this article, that gave valuable remarks, comments, and advices, in order to revise and improve the results of the paper.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

Funding
Faculty of Science, Damietta University, New Damietta, Egypt.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Author details
[1]Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt. [2]Department of Mathematics, Faculty of Science and Arts in Badaya, Al-Qassim University, Al- Badaya, Al-Qassim, Kingdom of Saudi Arabia. [3]Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania.

Received: 22 May 2019 Accepted: 14 October 2019
Published online: 07 November 2019

References

1. Bulboacă, T.: Differential subordinations and superordinations. Recent Results, House of Scientific Book Publ., Cluj-Napoca (2005)
2. Miller, S. S., Mocanu, P. T.: Differential subordinations: theory and applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225. Marcel Dekker Inc., New York and Basel (2000)
3. Szász, R., Kupán, P. A.: About the univalence of the Bessel functions, Studia Univ. Babeş-Bolyai Math. **54**(1), 127–132 (2009)
4. Baricz, Á.: Geometric propertis of generalized Bessel functions. Publ. Math. Debr. **73**, 155–178 (2008)
5. Jackson, F. H.: The application of basic numbers to Bessel's and Legendre's functions. Proc. Lond. Math. Soc. **3**(2), 1–23 (1905)
6. Selvakumaran, K. A., Szász, R.: Certain geometric properties of an integral operator involving Bessel functions. Kyungpook Math. J. **58**, 507–517 (2018)
7. Carathéodory, C.: Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen. Math. Ann. **64**, 95–115 (1907)
8. Libera, R. J., Zlotkiewicz E.J.: Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$. Proc. Amer. Math. Soc. **87**(2), 251–257 (1983)

9. Ma, W., Minda, D.: A unified treatment of some special classes of univalent functions. In: Li, Z., Ren, F., Lang, L., Zhang, S. (eds.) Proceedings of the Conference on Complex Analysis, Tianjin, 1992, pp. 157–169. Conf. Proc. Lecture Notes Anal. I, Int. Press, Cambridge, (1994)
10. Porwal, S.: An application of a Poisson distribution series on certain analytic functions. J. Complex Anal., 1–3 (2014). Art. ID 984135. https://doi.org/10.1155/2014/984135
11. Srivastava, D., Porwal, S.: Some sufficient conditions for Poisson distribution series associated with conic regions. Int. J. Adv. Technol. Eng. Sci. **3**(1), 229–235 (2015)

Publisher's Note