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Oscillation of linear third-order impulsive difference equations with variable coefficients

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Abstract

The present work discusses the qualitative behaviour of solutions of third-order difference equations of the form:

$$w(l+3)+a(l)w(l+2)+b(l)w(l+1)+c(l)w(l)=0,\ l\neq\theta_k,\ l\geq l_0$$

subject to the impulsive condition

$$w(\theta_k)=\alpha_k w(\theta_k-1),\ k\in\mathbb{N}.$$

Our state of the art is the inequality technique under the control of fixed moments of impulsive effect. We give some numerical examples to illustrate our findings.

Keywords: Oscillation, Nonoscillation, Third-order difference equation, Discrete impulse

Mathematics Subject Classification: 39A10, 39A12, 39A21

Introduction

In many applications, differential equations with impulses play a vital role [11, 15, 16, 20]. However, many scholars have recently taken an interest in studying discrete-time systems since both continuous-time systems and discrete-time systems share the same importance in theory as well as application. More specifically, it is required to study the corresponding discrete-time system when studying a continuous-time system, as discrete-time systems are easier to handle on computers than their continuous-time counterparts [21]. In particular, Li et al. [12] have investigated the oscillatory behaviour of a type of third-order difference equations with impulse of the form:

$$(E_1)\begin{cases} \Delta^3\mathcal{V}(\eta)+p(\eta)\mathcal{V}(\eta-\tau)=0,\ \eta\neq\eta_k\\ \mathcal{V}(\eta_k)=a_k\mathcal{V}(\eta_k-1)),\ k\in\mathbb{N}\\ \Delta\mathcal{V}(\eta_k)=b_k\Delta\mathcal{V}(\eta_k-1),\ k\in\mathbb{N}\\ \Delta^2\mathcal{V}(\eta_k)=c_k\Delta^2\mathcal{V}(\eta_k-1),\ k\in\mathbb{N}\end{cases}$$

and it is not a surprise to see the extension work in [13] for nonlinear third-order difference equations

$$(E_2)\begin{cases} \Delta^3\mathcal{V}(\eta)+p(\eta)F(\mathcal{V}(\eta-\tau))=0,\ \ \eta\neq\eta_k\\ \Delta^i\mathcal{V}(\eta_k)=g_{i,k}\left(\Delta^i\mathcal{V}(\eta_k-1)\right),\ \ i=0,1,2,\ \ k\in\mathbb{N},\end{cases}$$

where $a_{i,k}\leq\frac{g_{i,k}(u)}{u}\leq b_{i,k}$. It needs to be noted that the works [12, 13] are strict followers of the pioneer works [26, 14] in which the authors have studied

$$(E_3)\begin{cases} \mathcal{V}'''(t)+p(t)\mathcal{V}(t)=0,\ t\geq t_0,\ t\neq t_k\\ \mathcal{V}(t_k^+)=a_k\mathcal{V}(t_k),\ \mathcal{V}'(t_k^+)=b_k\mathcal{V}'(t_k),\ \mathcal{V}''(t_k^+)=c_k\mathcal{V}''(t_k),\ k\in\mathbb{N}\\ \mathcal{V}(t_0^+)=\mathcal{V}(t_0),\ \mathcal{V}'(t_0^+)=\mathcal{V}'(t_0),\ \mathcal{V}''(t_0^+)=\mathcal{V}''(t_0)\end{cases}$$

and

$$(E_4)\begin{cases} \mathcal{V}'''(t)+f(t,\mathcal{V}(t-\tau))=0,\ t\geq t_0,\ t\neq t_k\\ \mathcal{V}(t_k^+)=g_k(\mathcal{V}(t_k)),\ \mathcal{V}'(t_k^+)=h_k(\mathcal{V}'(t_k)),\ \mathcal{V}''(t_k^+)=l_k(\mathcal{V}''(t_k)),\ k\in\mathbb{N}\ ,\\ \mathcal{V}(t)=\phi(t),\ t\in[t_0-\tau,t_0]\end{cases}$$

respectively. We note that (E_2) is the discrete analogue of (E_4) (simultaneously (E_3)).

The third-order delay difference equation in (E_1) is a special case of the third-order difference equation

$$w(l+3)+a(l)w(l+2)+b(l)w(l+1)+c(l)w(l)=0,\ l\geq l_0 \tag{1.1}$$

when $\tau=0$. If $\tau\neq 0$, then the equation in (E_1) can be reduced to a higher order difference equation. It is not known whether the work [12] will apply to (1.1) if we need to attach the impulsive effect

$$w(\theta_k)=\alpha_k w(\theta_k-1),\ k\in\mathbb{N}. \tag{1.2}$$

Based on the motivation stated above, the goal of this paper is to discuss the oscillatory behaviour of solutions of the impulsive system

$$(\mathcal{E})\begin{cases} w(l+3)+a(l)w(l+2)+b(l)w(l+1)+c(l)w(l)=0,\ l\neq\theta_k,\ l\geq l_0\\ w(\theta_k)=\alpha_k w(\theta_k-1),\ k\in\mathbb{N},\end{cases}$$

where $a(l)$, $b(l)$ and $c(l)$ are real-valued functions with discrete arguments and we assume that there exists $\epsilon>0$ such that $\alpha_k\geq\epsilon>0$ for $k\in\mathbb{N}$. For our impulsive system, $\theta_1,\theta_2,\theta_3,\ldots$ are discrete moments of impulsive effect with the properties $0\leq l_0<\theta_1<\theta_2<\cdots<\theta_k$ and $\lim_{k\to\infty}\theta_k=+\infty$. By a solution of (1.1), we mean a real-valued function $w(l)$ defined on $l\geq l_0$ which satisfy (1.1) for $l\geq l_0$. If $w(l_0),w(l_0+1)$ and $w(l_0+2)$ are given, then (1.1) admits a unique solution for $l\geq l_0$. A solution $w(l)$ of $(\mathcal{E})$ is said to be oscillatory if for every $n\geq 0$, there exists $l\geq n$ such that $w(l+1)w(l)\leq 0$; otherwise, the solution is said to be nonoscillatory. The system $(\mathcal{E})$ is oscillatory if all its solutions are oscillatory.

For more information on qualitative studies of discrete impulsive equations, see the work [6] in which Danca et al. studied a kind of impulsive equations

$$\begin{cases} \mathcal{V}(n+1) = f(\mathcal{V}(n)),\ n \neq n_i \\ \mathcal{V}(n_i+1) = f(g(\mathcal{V}(n_i))),\ i \in \mathbb{N} \end{cases}$$

for the existence of periodic orbits, asymptotic behaviour, and chaos. In another work [7], Danca and Feckan studied the chaotic behaviour of a one-dimensional discrete supply and demand impulsive dynamical system

$$\mathcal{V}(n+1) = \begin{cases} f(\mathcal{V}(n)),\ n \neq n_i \\ \mathcal{V}(n) + \gamma,\ n = n_i. \end{cases}$$

We refer the reader to some of the monographs [1, 2, 8, 10, 11, 15, 16, 20] and works [3–5, 9, 22] for more detailed treatments on the theory of impulsive difference equations and its applications.

Methods

The first method that has been adopted here is the method of contradiction. By making use of suitable impulsive inequalities, we have seen that the problem $(\mathcal{E})$ under investigation does not have any nonoscillatory solutions. Once this is done, there is a conclusion. Secondly, we generate two linear operators from a particular set of linearly independent solutions of $(\mathcal{E})$. We get the results by using matrix theory and the definition of nonoscillatory solutions of $(\mathcal{E})$.

Results

In this section, we discuss the oscillatory behaviour of solutions of the discrete impulsive system of third order of the form $(\mathcal{E})$.

Theorem 3.1 *Let $a(l) < 0, c(l) \geq 0$ and $b(l) \geq 0$ for large l. Assume that*

$\liminf_{l\to\infty} b(l) = b \geq 0$. *If*

$$(H_1)\quad \limsup_{k\to\infty}\left[1 + \frac{\alpha_k}{a(\theta_k - 3)}\right] > 0,$$

$$(H_2)\quad \limsup_{k\to\infty}\left[\alpha_k + \frac{b}{a(\theta_k - 2)}\right] > 0$$

and

$$(H_3)\quad \limsup_{k\to\infty} b(\theta_k) > \limsup_{k\to\infty} a(\theta_k - 1)\left[a(\theta_k) - \frac{b}{a(\theta_k+1)}\right]$$

hold, then $(\mathcal{E})$ is oscillatory.

Proof

Suppose that w(l) is a nonoscillatory solution of $(\mathcal{E})$. So, there exists $L_1 > 0$ such that $w(l) > 0$ for $l \geq L_1$. For $l \neq \theta_k$, it follows from $(\mathcal{E})$ that

$$-a(l)w(l+2) \geq w(l+3),$$

that is,

$$w(l+2) \geq \frac{-w(l+3)}{a(l)} \quad \text{for } l \geq L_1.$$

For $l \geq L_1 + 1$, it follows that

$$w(l+1) \geq \frac{-w(l+2)}{a(l-1)}. \tag{3.1}$$

Again from $(\mathcal{E})$, we can find $L_2 > L_1$ such that

$$-a(l)w(l+2) > b(l)w(l+1) > bw(l+1), l \geq L_2$$

for which

$$w(l+2) > \frac{-b}{a(l)}w(l+1) \quad \text{for } l \geq L_2. \tag{3.2}$$

Let $L_3 = \max\{L_1 + 1, L_2\}$. We claim that (3.1) hold for $l = \theta_k$, that is,

$$w(\theta_k + 1) \geq \frac{-w(\theta_k + 2)}{a(\theta_k - 1)} \quad \text{for } \theta_k \geq L_3,$$

which is equivalent to

$$w(\theta_k - 1) \geq \frac{-w(\theta_k)}{a(\theta_k - 3)} \quad \text{for } \theta_k \geq L_3 + 2.$$

If possible, there is some $j > k$ such that

$$w(\theta_j - 1) < \frac{-w(\theta_j)}{a(\theta_j - 3)} = \frac{-\alpha_j w(\theta_j - 1)}{a(\theta_j - 3)},$$

that is,

$$w(\theta_j - 1)\Big[1 + \frac{\alpha_j}{a(\theta_j - 3)}\Big] < 0, \quad \theta_j \geq L_3 + 2,$$

a contradiction to (H_1). So, (3.1) holds for all $\theta_k \geq L_3 + 2$. Again, we have another claim that (3.2) hold for $l = \theta_k$, that is,

$$w(\theta_k + 2) > \frac{-b}{a(\theta_k)}w(\theta_k + 1) \quad \text{for } \theta_k \geq L_3,$$

which is equivalent to

$$w(\theta_k) > \frac{-b}{a(\theta_k - 2)}w(\theta_k - 1) \quad \text{for } \theta_k \geq L_3 + 2.$$

If not, let there exist $i > k$ such that

$$w(\theta_i) \leq \frac{-b}{a(\theta_i - 2)} w(\theta_i - 1),$$

that is,

$$\alpha_i w(\theta_i - 1) = w(\theta_i) \leq \frac{-b}{a(\theta_i - 2)} w(\theta_i - 1).$$

Consequently,

$$w(\theta_i - 1)\left[\alpha_i + \frac{b}{a(\theta_i - 2)}\right] \leq 0, \theta_i \geq L_3 + 2,$$

a contradiction to (H_2). Thus, our claim holds for all $\theta_k \geq L_3 + 2$. Let $L \geq L_3 + 2$. For $\theta_k \geq L$, we have

$$\begin{aligned} 0 &\geq w(\theta_k + 3) + a(\theta_k)w(\theta_k + 2) + b(\theta_k)w(\theta_k + 1) + \alpha_k c(\theta_k)w(\theta_k - 1) \\ &\geq \frac{-b}{a(\theta_k + 1)} w(\theta_k + 2) + a(\theta_k)w(\theta_k + 2) - \frac{b(\theta_k)w(\theta_k + 2)}{a(\theta_k - 1)}. \end{aligned}$$

As a result,

$$\frac{-b}{a(\theta_k + 1)} + a(\theta_k) - \frac{b(\theta_k)}{a(\theta_k - 1)} \leq 0,$$

that is,

$$b(\theta_k) \leq -a(\theta_k - 1)\left[\frac{b}{a(\theta_k + 1)} - a(\theta_k)\right],$$

a contradiction to (H_3). Because (3.1) and (3.2) are true for all l and $\theta_k, k \in \mathbb{N}$, then (H_3) holds true for l also. This is all about the proof of the theorem. $\square$

Example 3.2
Consider

$$\begin{cases} w(l+3) - 5w(l+2) + 9(2 + \cos\frac{l\pi}{3})w(l+1) + 2w(l) = 0, l \neq \theta_k \\ w(\theta_k) = \alpha_k w(\theta_k - 1), k \in \mathbb{N}, \end{cases}$$

where $a(l) = -5$, $b(l) = 9(2 + \cos\frac{l\pi}{3})$, $c(l) = 2$, $\theta_k = 3k + 3$, $k \in \mathbb{N}$ and $\alpha_k = 2 + \frac{1}{k+1}$. Indeed, $\liminf_{l\to\infty} b(l) = 9 > 0$. From (H_1), (H_2) and (H_3), we get

$$\limsup_{k\to\infty}\left(1 - \frac{2k+2}{5k+5}\right) = \frac{3}{5} > 0,$$
$$\limsup_{k\to\infty}\left(\frac{1}{1+k} - \frac{1}{5}\right) = \frac{1}{5} > 0,$$

and

$$27 = \limsup_{k\to\infty} b(\theta_k) > \limsup_{k\to\infty} a(\theta_k - 1)\left(a(\theta_k) - \frac{b}{a(\theta_k + 1)}\right) = 16,$$

respectively. Clearly, all conditions of Theorem 3.1 are satisfied for the given system. Hence, the system is oscillatory.

In particular, consider the nonimpulsive difference equation

$$w(l+3) - 5w(l+2) + 18w(l+1) + 2w(l) = 0, \quad l \geq l_0. \tag{3.3}$$

It is not difficult to see that (3.3) is oscillatory because (H_3) holds, that is, $a^2 < 2b$. A graphical illustration is given in Fig. 1. So, it remains oscillating after the imposition of proper impulse controls.

Theorem 3.3 *Let $a(l) > 0$, $b(l) < 0$ and $c(l) \geq 0$ for large l. Assume that*

$\liminf_{l\to\infty} c(l) = c \geq 0$. *If*

$$(H_4) \quad \limsup_{k\to\infty} \left[1 + \frac{\alpha_k a(\theta_k - 2)}{b(\theta_k - 2)}\right] > 0,$$

$$(H_5) \quad \limsup_{k\to\infty} \left[\alpha_k + \frac{c}{b(\theta_k - 1)}\right] > 0$$

and

$$(H_6) \quad \limsup_{k\to\infty} c(\theta_k) > \limsup_{k\to\infty} \frac{b(\theta_k - 1)}{a(\theta_k - 1)}\left[b(\theta_k) - \frac{ca(\theta_k)}{b(\theta_k + 1)}\right]$$

hold, then $(\mathcal{E})$ oscillatory.

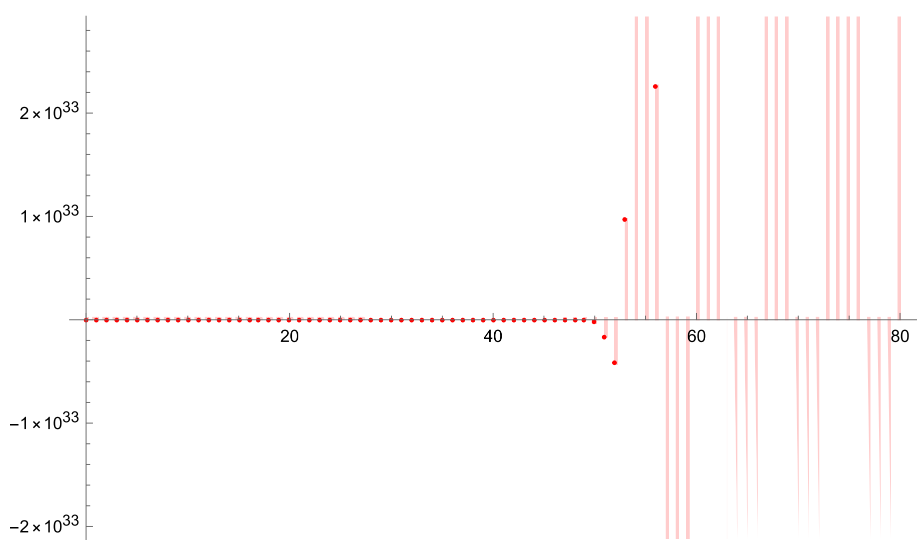

Fig. 1 Solution of Eq. (3.3) for $l \in (0, 80)$ with initial conditions $w(l_0) = 1$, $w(l_0 + 1) = 3$ and $w(l_0 + 2) = 4$

Proof

Suppose on the contrary, $w(l)$ is a nonoscillatory solution of $(\mathcal{E})$. So, there exists $L_1 > 0$ such that $w(l) > 0$ for $l \geq L_1$. For $l \neq \theta_k$, it follows from $(\mathcal{E})$ that

$$-b(l)w(l+1) > a(l)w(l+2),$$

that is,

$$w(l+1) > \frac{-a(l)}{b(l)}w(l+2),\ l \geq L_1,$$

which is equivalent to

$$w(l) > \frac{-a(l-1)}{b(l-1)}w(l+1) \quad \text{for } l \geq L_1 + 1. \tag{3.4}$$

Also, from $(\mathcal{E})$ we have for $l \geq L_2$

$$-b(l)w(l+1) > c(l)w(l) > cw(l)$$

which reduces to

$$w(l+1) > \frac{-c}{b(l)}w(l),\ l \geq L_2. \tag{3.5}$$

Let $L_3 = \max\{L_1+1, L_2\}$. We claim that (3.4) holds for $l = \theta_k$, that is,

$$w(\theta_k) > \frac{-a(\theta_k-1)}{b(\theta_k-1)}w(\theta_k+1) \quad \text{for } \theta_k \geq L_3,$$

that is,

$$w(\theta_k-1) > \frac{-a(\theta_k-2)}{b(\theta_k-2)}w(\theta_k) \quad \text{for } \theta_k \geq L_3+1.$$

If not, let there exist $j > k$ such that

$$w(\theta_j-1) \leq \frac{-a(\theta_j-2)}{b(\theta_j-2)}w(\theta_j) = \frac{-\alpha_j a(\theta_j-2)}{b(\theta_j-2)}w(\theta_j-1),$$

that is,

$$w(\theta_j-1)\left[1+\frac{\alpha_j a(\theta_j-2)}{b(\theta_j-2)}\right] \leq 0,\ \theta_j \geq L_3+1,$$

a contradiction to (H_4). So, our claim holds for all $\theta_k \geq L_3+1$. Next, we claim that (3.5) holds for $l = \theta_k$, that is,

$$w(\theta_k+1) > \frac{-c}{b(\theta_k)}w(\theta_k) \quad \text{for } \theta_k \geq L_3,$$

which is equivalent to

$$w(\theta_k) > \frac{-c}{b(\theta_k - 1)} w(\theta_k - 1) \quad \text{for } \theta_k \geq L_3 + 1.$$

Otherwise, there exists $i > k$ such that

$$w(\theta_i) \leq \frac{-c}{b(\theta_i - 1)} w(\theta_i - 1),$$

that is,

$$\alpha_i w(\theta_i - 1) = w(\theta_i) \leq \frac{-c}{b(\theta_i - 1)} w(\theta_i - 1).$$

As a result,

$$w(\theta_i - 1)\left[\alpha_i + \frac{c}{b(\theta_i - 1)}\right] \leq 0, \theta_i \geq L_3 + 1,$$

a contradiction to (H_5). Thus, (3.5) holds true for all $\theta_k \geq L_3 + 1$. Let $L = L_3 + 1$. For $\theta_k \geq L$, we have

$$\begin{aligned} 0 &> a(\theta_k)w(\theta_k + 2) + b(\theta_k)w(\theta_k + 1) + c(\theta_k)w(\theta_k) \\ &> \frac{-ca(\theta_k)}{b(\theta_k + 1)} w(\theta_k + 1) + b(\theta_k)w(\theta_k + 1) - \frac{c(\theta_k)a(\theta_k - 1)}{b(\theta_k - 1)} w(\theta_k + 1), \end{aligned}$$

that is,

$$\frac{-ca(\theta_k)}{b(\theta_k + 1)} + b(\theta_k) - \frac{c(\theta_k)a(\theta_k - 1)}{b(\theta_k - 1)} < 0,$$

a contradiction to (H_6). Since (3.4) and (3.5) are true for all l and $\theta_k, k \in \mathbb{N}$, then (H_6) holds true for l also. Hence, the theorem is proved. □

Example 3.4
Consider

$$\begin{cases} w(l+3) + a(l)w(l+2) - 4w(l+1) + c(l)w(l) = 0, l \neq \theta_k \\ w(\theta_k) = \alpha_k w(\theta_k - 1), k \in \mathbb{N}, \end{cases}$$

where $a(l) = 2 + \frac{1}{2l}$, $b(l) = -4$, $c(l) = 5\left(1 + \frac{1}{l^2}\right)$, $\theta_k = 2k + 3$, $k \in \mathbb{N}$ and $\alpha_k = \frac{3k}{2(k+1)} = 1 + \frac{k-2}{2(k+1)}$. Indeed, $\liminf_{l \to \infty} c(l) = 5$. From (H_4), (H_5) and (H_6), we get

$$\limsup_{k \to \infty} \left(1 - \frac{24k^2 + 15k}{32k^2 + 48k + 16}\right) = \frac{1}{4} > 0,$$

$$\limsup_{k \to \infty} \left(\frac{3k}{2k+2} - \frac{5}{4}\right) = \frac{1}{4} > 0,$$

and

$$5 = \limsup_{k\to\infty} c(\theta_k) > \limsup_{k\to\infty} \left(\frac{16(4k+4)}{(8k+9)} - \frac{(4k+4)(40k+65)}{(4k+6)(8k+9)} \right) = 3,$$

respectively. Therefore, all conditions of Theorem 3.3 are satisfied here for the given system. Thus, the system is oscillatory.

In particular, consider the nonimpulsive difference equation

$$w(l+3) + 2w(l+2) - 4w(l+1) + 5w(l) = 0, \quad n \geq n_0. \tag{3.6}$$

It is not difficult to see that (3.6) is oscillatory because (H_6) holds, that is, $b^2 < 2ac$. A graphical illustration is given in Fig. 2. So, it remains oscillating after the imposition of proper impulse controls.

Theorem 3.5 *Let $a(l) \geq 0$, $b(l) < 0$ and $c(l) \geq 0$ for large l. Assume that*

$$(H_7) \quad \limsup_{k\to\infty} \left[1 + \frac{\alpha_k a(\theta_k - 2)}{b(\theta_k - 2)} \right] > 0,$$

$$(H_8) \quad \limsup_{k\to\infty} \left[\alpha_k + \frac{c(\theta_k - 1)}{b(\theta_k - 1)} \right] > 0$$

and

$$(H_9) \quad \limsup_{k\to\infty} b(\theta_k) > \limsup_{k\to\infty} \left[\frac{a(\theta_k)c(\theta_k+1)}{b(\theta_k+1)} + \frac{a(\theta_k - 1)c(\theta_k)}{b(\theta_k - 1)} \right]$$

hold. Then, $(\mathcal{E})$ is oscillatory.

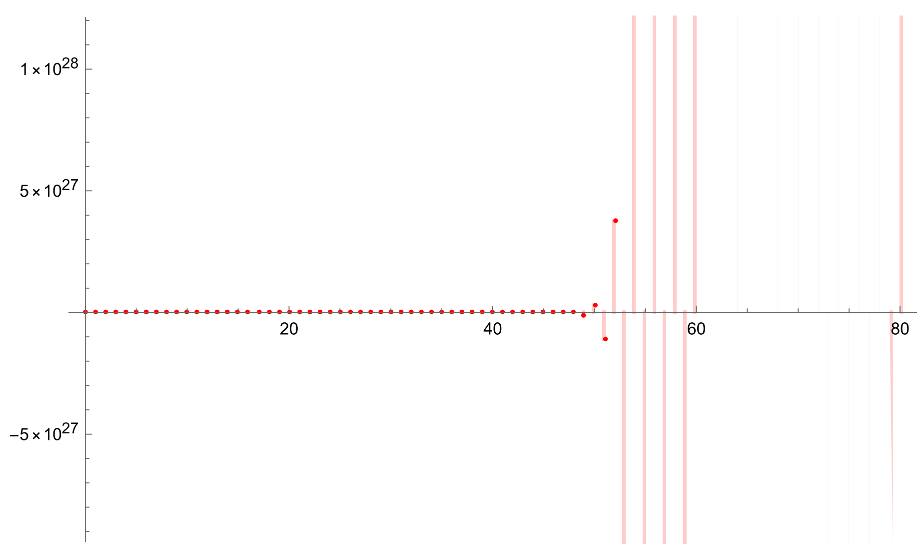

Fig. 2 Solution of Eq. (3.6) for $l \in (0, 80)$ with initial conditions $w(l_0) = 2, w(l_0+1) = 1$ and $w(l_0+2) = 1$

Proof

On contrary, we assume that w(l) is a nonoscillatory solution of $(\mathcal{E})$. *Then, there exists* $L_1 > 0$ *such that* $w(l) > 0$ *for* $l \geq L_1$. *From* $(\mathcal{E})$ *and for* $l \neq \theta_k$, *we have*

$$-b(l)w(l+1) > a(l)w(l+2),$$

that is,

$$w(l+1) > \frac{-a(l)}{b(l)}w(l+2) \quad \text{for } l \geq L_1. \tag{3.7}$$

Also for $l \geq L_2$, we have that

$$-b(l)w(l+1) > c(l)w(l),$$

that is,

$$w(l+1) > \frac{-c(l)}{b(l)}w(l) \quad \text{for } l \geq L_2 \tag{3.8}$$

Let $L_3 = \max\{L_1, L_2\}$. Now, we claim that (3.7) hold for $l = \theta_k$, that is,

$$w(\theta_k + 1) > \frac{-a(\theta_k)}{b(\theta_k)}w(\theta_k + 2) \quad \text{for } \theta_k \geq L_3,$$

that is,

$$w(\theta_k - 1) > \frac{-a(\theta_k - 2)}{b(\theta_k - 2)}w(\theta_k) \quad \text{for } \theta_k \geq L_3 + 2.$$

If not, there exists $j > k$ such that

$$w(\theta_j - 1) \leq \frac{-a(\theta_j - 2)}{b(\theta_j - 2)}w(\theta_j) = \frac{-\alpha_j a(\theta_j - 2)}{b(\theta_j - 2)}w(\theta_j - 1)$$

and hence

$$w(\theta_j - 1)\left[1 + \frac{\alpha_j a(\theta_j - 2)}{b(\theta_j - 2)}\right] \leq 0, \ \theta_j \geq L_3 + 2,$$

a contradiction to (H_7). Therefore, our claim holds. Also, we have another claim that (3.8) holds for $i = \theta_k$, that is,

$$w(\theta_k + 1) > \frac{-c(\theta_k)}{b(\theta_k)}w(\theta_k) \quad \text{for } \theta_k \geq L_3$$

which is equivalent to

$$w(\theta_k) > \frac{-c(\theta_k - 1)}{b(\theta_k - 1)}w(\theta_k - 1) \quad \text{for } \theta_k \geq L_3 + 1.$$

Suppose there exists $i > k$ such that

$$w(\theta_i) \leq \frac{-c(\theta_i-1)}{b(\theta_i-1)} w(\theta_i-1),$$

that is,

$$\alpha_i w(\theta_i-1) = w(\theta_i) \leq \frac{-c(\theta_i-1)}{b(\theta_i-1)} w(\theta_i-1).$$

Consequently,

$$w(\theta_i-1)\left[\alpha_i + \frac{c(\theta_i-1)}{b(\theta_i-1)}\right] \leq 0 \quad \text{for } \theta_i \geq L_3+1$$

gives a contradiction to (H_8). Thus, (3.8) holds for all $\theta_k \geq L_3+1$. Let $L = L_3+1$. For all $\theta_k \geq L$, we have

$$\begin{aligned} 0 &> a(\theta_k)w(\theta_k+2) + b(\theta_k)w(\theta_k+1) + c(\theta_k)w(\theta_k) \\ &> \frac{-a(\theta_k)c(\theta_k+1)}{b(\theta_k+1)} w(\theta_k+1) + b(\theta_k)w(\theta_k+1) - \frac{c(\theta_k)a(\theta_k-1)}{b(\theta_k-1)} w(\theta_k+1) \end{aligned}$$

implies that

$$\frac{-a(\theta_k)c(\theta_k+1)}{b(\theta_k+1)} + b(\theta_k) - \frac{c(\theta_k)a(\theta_k-1)}{b(\theta_k-1)} < 0$$

$\theta_k \geq L$, a contradiction to (H_9). This is all about the proof. □

Theorem 3.6 *Let $a(l) < 0$, $b(l) \geq 0$ and $c(l) \geq 0$ for large l. Assume that*

$$(H_{10}) \ \limsup_{k\to\infty}\left[1 + \frac{\alpha_k}{a(\theta_k-3)}\right] > 0,$$

$$(H_{11}) \ \limsup_{k\to\infty}\left[\alpha_k + \frac{b(\theta_k-2)}{a(\theta_k-2)}\right] > 0$$

and

$$(H_{12}) \ \limsup_{k\to\infty} \frac{c(\theta_k+1)}{a(\theta_k+1)a(\theta_k-1)} > \limsup_{k\to\infty}\left[\frac{b(\theta_k+1)}{a(\theta_k+1)} + \frac{b(\theta_k)}{a(\theta_k-1)} - a(\theta_k)\right]$$

hold. Then, $(\mathcal{E})$ is oscillatory.

Proof

On contrary, let $w(l)$ be a nonoscillatory solution of $(\mathcal{E})$. Then, there exists $L_1 > 0$ such that $w(l) > 0$ for $l \geq L_1$. From $(\mathcal{E})$ and for $l \neq \theta_k$, we have

$$-a(l)w(l+2) \geq w(l+3),$$

that is,

$$w(l+1) > \frac{-w(l+2)}{a(l-1)} \quad \text{for } n \geq L_1+1. \tag{3.9}$$

Similarly, we can find $L_2 > 0$ such that

$$-a(l)w(l+2) \geq c(l)w(l) + b(l)w(l+1),\ l \geq L_2,$$

that is,

$$w(l+3) \geq \frac{-c(l+1)}{a(l+1)}w(l+1) - \frac{b(l+1)}{a(l+1)}w(l+2) \quad \text{for } l \geq L_2 \tag{3.10}$$

Let $L_3 = \max\{L_1+1, L_2\}$. We claim that (3.9) hold for $l = \theta_k$, that is,

$$w(\theta_k+1) > \frac{-w(\theta_k+2)}{a(\theta_k-1)} \quad \text{for } \theta_k \geq L_3,$$

that is,

$$w(\theta_k-1) > \frac{-w(\theta_k)}{a(\theta_k-3)} \quad \text{for } \theta_k \geq L_3+2.$$

If not, there exists $j > k$ such that

$$w(\theta_j-1) \leq \frac{-w(\theta_j)}{a(\theta_j-3)} = \frac{-\alpha_j w(\theta_j-1)}{a(\theta_j-3)},$$

that is,

$$w(\theta_j-1)\left[1+\frac{\alpha_j}{a(\theta_j-3)}\right] \leq 0,\ \theta_j \geq L_3+2,$$

a contradiction to (H_{10}). Therefore, (3.9) holds for all $\theta_k \geq L_3$. Next, we claim that (3.10) holds for $l = \theta_k$, that is,

$$w(\theta_k+3) > \frac{-c(\theta_k+1)}{a(\theta_k+1)}w(\theta_k+1) - \frac{b(\theta_k+1)}{a(\theta_k+1)}w(\theta_k+2) \quad \text{for } \theta_k \geq L_3,$$

that is,

$$w(\theta_k) > \frac{-c(\theta_k-2)}{a(\theta_k-2)}w(\theta_k-2) - \frac{b(\theta_k-2)}{a(\theta_k-2)}w(\theta_k-1) > -\frac{b(\theta_k-2)}{a(\theta_k-2)}w(\theta_k-1)$$

for $\theta_k \geq L_3+3$. Suppose the claim is not true. Then, there exists $i > k$ such that

$$w(\theta_i) \leq -\frac{b(\theta_i-2)}{a(\theta_i-2)}w(\theta_i-1),$$

that is,

$$\alpha_i w(\theta_i-1) \leq \frac{-b(\theta_i-2)}{a(\theta_i-2)}w(\theta_i-1).$$

As a result,

$$w(\theta_i-1)\left[\alpha_i+\frac{b(\theta_i-2)}{a(\theta_i-2)}\right] \leq 0 \quad \text{for } \theta_i \geq L_3+3,$$

a contradiction to (H_{11}). Thus, (3.10) holds true for all $\theta_k \geq L_3 + 3$. Let $L = L_3 + 3$. Therefore, from $(\mathcal{E})$ and for all $\theta_k \geq L$, we see that

$$\begin{aligned} 0 &> w(\theta_k + 3) + a(\theta_k)w(\theta_k + 2) + b(\theta_k)w(\theta_k + 1) \\ &> \left[\frac{-b(\theta_k + 1)}{a(\theta_k + 1)} + \frac{c(\theta_k + 1)}{a(\theta_k + 1)a(\theta_k - 1)}\right]w(\theta_k + 2) + a(\theta_k)w(\theta_k + 2) \\ &- \frac{b(\theta_k)}{a(\theta_k - 1)}w(\theta_k + 2). \end{aligned}$$

Consequently,

$$\frac{-b(\theta_k + 1)}{a(\theta_k + 1)} + \frac{c(\theta_k + 1)}{a(\theta_k + 1)a(\theta_k - 1)} + a(\theta_k) - \frac{b(\theta_k)}{a(\theta_k - 1)} < 0$$

for $\theta_k \geq L$, a contradiction to (H_{12}). This is all about the proof. □

Example 3.7
Consider

$$\begin{cases} w(l + 3) - 3w(l + 2) + 2w(l + 1) + 12lw(l) = 0, l \neq \theta_k \\ w(\theta_k) = \alpha_k w(\theta_k - 1), k \in \mathbb{N}, \end{cases}$$

where $a(l) = -3$, $b(l) = 2$, $c(l) = 12l$, $\theta_k = 3^k$, $k \in \mathbb{N}$ and $\alpha_k = 2 + \frac{1}{3^k}$. From (H_{10}), (H_{11}) and (H_{12}), we get

$$\limsup_{k\to\infty}\left(1 - \frac{2 \times 3^k + 1}{3 \times 3^k}\right) = \frac{1}{3} > 0,$$
$$\limsup_{k\to\infty}\left(2 - \frac{2}{3} + \frac{1}{3^k}\right) = \frac{4}{3} > 0,$$

and

$$\frac{5}{3} = \limsup_{k\to\infty}\left[\frac{b(\theta_k + 1)}{a(\theta_k + 1)} + \frac{b(\theta_k)}{a(\theta_k - 1)} - a(\theta_k)\right] < \limsup_{k\to\infty}\frac{c(\theta_k + 1)}{a(\theta_k + 1)a(\theta_k - 1)} \to \infty,$$

respectively. Theorem 3.6 is applicable to the given system, and showing that it is oscillatory.

In particular, consider the nonimpulsive difference equation

$$w(l + 3) - 3w(l + 2) + 2w(l + 1) + 12w(l) = 0, \quad n \geq n_0. \tag{3.11}$$

It is not difficult to see that (3.11) is oscillatory because (H_{12}) holds, that is, $c > 2ab - a^3$. A graphical illustration is given in Fig. 3. So, it remains oscillating after the imposition of proper impulse controls.

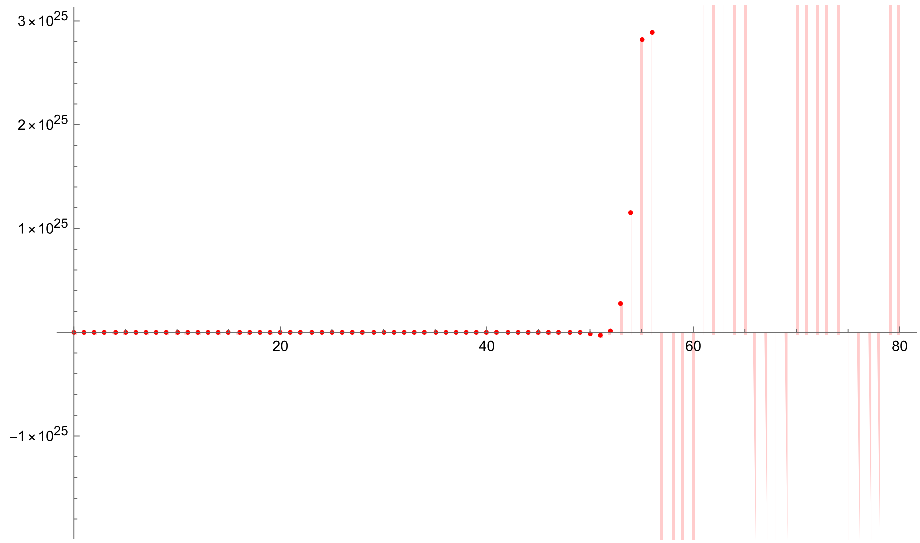

Fig. 3 Solution of Eq. (3.11) for $l \in (0, 80)$ with initial conditions $w(l_0) = 1$, $w(l_0 + 1) = 4$ and $w(l_0 + 2) = 2$

Next, we are going to present some new results, and for this, we need the following lemma.

Lemma 3.8 *Suppose there exists $l_0 \geq 0$ such that $a(l) < 0$, $b(l) < 0$ and $c(l) < 0$ for $l \geq l_0$. If $w(l)$ is a solution of*

$$w(l+3) + a(l)w(l+2) + b(l)w(l+1) + c(l)w(l) = 0$$

with the initial conditions $w(l_0) \geq 0$, $w(l_0+1) \geq 0$, $w(l_0+2) \geq 0$ such that $w(l_0) + w(l_0+1) + w(l_0+2) > 0$ for some $l_0 \geq 0$, then $w(l) > 0$ for $l \geq l_1 = l_0 + 3$.

Proof
Upon the application of initial conditions to

$$w(l+3) = -a(l)w(l+2) - b(l)w(l+1) - c(l)w(l),$$

it is easy to see recursively that $w(l) > 0$ for $l \geq l_1 = l_0 + 3$ when $l \neq \theta_k$. Assume that $l = \theta_k$ for $l \in \mathbb{N}$. Let $\theta_1 \geq l_1 + 1$. Then, $w(\theta_1) = \alpha_1 w(\theta_1 - 1) > 0$. Since $\theta_2 - 1$ and $\theta_3 - 1$ are nonimpulsive points, then $w(\theta_2) = \alpha_2 w(\theta_2 - 1) > 0$ and $w(\theta_3) = \alpha_3 w(\theta_3 - 1) > 0$. Due to

$$w(\theta_{k+3}) = -a(\theta_k)w(\theta_{k+2}) - b(\theta_k)w(\theta_{k+1}) - c(\theta_k)w(\theta_k),$$

and proceeding recursively, it is easy to see that $w(\theta_k) > 0$ for $k \in \mathbb{N}$. Hence, the lemma is proved. □

Theorem 3.9 *Let $a(l) < 0$, $b(l) < 0$ and $c(l) < 0$ for $l \geq l_0$. Then, $(\mathcal{E})$ admits two oscillatory solutions.*

Proof

Let us assume that $u^0(l)$, $u^1(l)$ and $u^2(l)$ be the solutions of (1.1) such that

$$u^i(l_0+j) = \begin{cases} 1 & i=j \\ 0 & i \neq j, \end{cases}$$

where $i,j = 0,1,2$ and $l_0 \geq 0$. Firstly, we consider the case when $l \neq \theta_k$. Indeed,

$$\begin{aligned} &u^0(l_0) = 1 \quad u^1(l_0) = 0 \quad u^2(l_0) = 0; \\ &u^0(l_0+1) = 0 \quad u^1(l_0+1) = 1 \quad u^2(l_0+1) = 0; \\ &u^0(l_0+2) = 0 \quad u^1(l_0+2) = 0 \quad u^2(l_0+2) = 1. \end{aligned}$$

As a result, the solutions resulting from $u^i(l_0+j)$ for $i,j = 0,1,2$ are linearly independent. According to Lemma 3.8, $u^i(l) > 0$ for $l \geq l_0+3$ and $i = 0,1,2$. It is possible to select nonzero real numbers d_{0n}, d_{2n}, e_{1n} and e_{2n} for each $n \geq l_0+3$ such that

$$\begin{aligned} &d_{0n}u^0(n) + d_{2n}u^2(n) = 0 \\ &e_{1n}u^1(n) + e_{2n}u^2(n) = 0, \end{aligned}$$

where $d_{0n}^2 + d_{2n}^2 = 1 = e_{1n}^2 + e_{2n}^2$. So, the preceding system is orthogonal. As a result, the solutions are linearly independent for $n \geq l_0+3$. For $l \geq l_0$, define

$$\begin{aligned} &T^n(l) = d_{0n}u^0(l) + d_{2n}u^2(l) \\ &\mathcal{T}^n(l) = e_{1n}u^1(l) + e_{2n}u^2(l). \end{aligned}$$

Then, $T^n(l)$ and $\mathcal{T}^n(l)$ are solutions of (1.1) with $T^n(n) = 0$ and $\mathcal{T}^n(n) = 0$. Since each of the sequence $\{d_{0n}\}$, $\{d_{2n}\}$, $\{e_{1n}\}$ and $\{e_{2n}\}$ is bounded, then there exists a $\{n_k\} \subset \{n\}$ such that

$$d_{0n_k} \to d_0, \quad d_{2n_k} \to d_2, \quad e_{1n_k} \to e_1, \quad e_{2n_k} \to e_2$$

as $n_k \to \infty$. Of course, $d_0^2 + d_2^2 = 1 = e_1^2 + e_2^2$. Setting

$$\begin{aligned} &T(l) = d_0u^0(l) + d_2u^2(l) \\ &\mathcal{T}(l) = e_1u^1(l) + e_2u^2(l), \end{aligned}$$

we notice that $T(l)$ and $\mathcal{T}(l)$ are nontrivial solutions of (1.1).

If $\{T(l)\}$ is nonoscillatory, then we may assume that $T(l) > 0$ for $l \geq L_1 > l_0+3$. For $0 < \epsilon_l < T(l)$, there exists $L_1 > 0$ such that

$$\left|T^{n_k}(l) - T(l)\right| < \epsilon_l \text{ for } n_k > L_l,$$

that is, $0 < T(l) - \epsilon_l < T^{n_k}(l)$ for $n_k \geq L_l$. Hence, for $n_k > \max\{L_1, L_l\}$, it follows that $T^{n_k}(n_k) > 0$, which is a contradiction to the fact that $T^{n_k}(n_k) = 0$. Thus, $T(l)$ is oscillatory. Similar argument we apply to $\mathcal{T}(l)$. Indeed, these two solutions are linearly independent.

Next, we consider the case when $l = \theta_k$. Since (1.1) takes to a similar mapping for $l = \theta_k$, then without loss of generality, we can assume an impulsive perturbation at each step. Using (1.2) in (1.1), we have

$$\frac{1}{\alpha_{k+3}} w(\theta_{k+3}) + \frac{a(\theta_k)}{\alpha_{k+2}} w(\theta_{k+2}) + \frac{b(\theta_k)}{\alpha_{k+1}} w(\theta_{k+1}) + \frac{c(\theta_k)}{\alpha_k} w(\theta_k) = 0,$$

that is,

$$w(\theta_{k+3}) + \frac{\alpha_{k+3}a(\theta_k)}{\alpha_{k+2}} w(\theta_{k+2}) + \frac{\alpha_{k+3}b(\theta_k)}{\alpha_{k+1}} w(\theta_{k+1}) + \frac{\alpha_{k+3}c(\theta_k)}{\alpha_k} w(\theta_k) = 0$$

which is of the form

$$w(\theta_{k+3}) + A(\theta_k)w(\theta_{k+2}) + B(\theta_k)w(\theta_{k+1}) + C(\theta_k)w(\theta_k) = 0, \tag{3.12}$$

where $A(\theta_k) = \frac{\alpha_{k+3}a(\theta_k)}{\alpha_{k+2}} < 0$, $B(\theta_k) = \frac{\alpha_{k+3}b(\theta_k)}{\alpha_{k+1}} < 0$ and $C(\theta_k) = \frac{\alpha_{k+3}c(\theta_k)}{\alpha_k} < 0$. Let us assume that $v^0(\theta_k)$, $v^1(\theta_k)$ and $v^2(\theta_k)$ are the solutions of (3.12) such that

$$v^i(\theta_{k_0+j}) = \begin{cases} 1 & i = j \\ 0 & i \neq j, \end{cases}$$

where $i, j = 0, 1, 2$ and $\theta_{k_0} \geq l_1 > l_0$. Proceeding as above, we can show that $T(\theta_k)$ and $\mathcal{T}(\theta_k)$ are oscillatory solutions; of course, these two solutions are linearly independent. Hence, our theorem is proved. □

Theorem 3.10 *Let $a(l) > 0$, $b(l) > 0$ and $c(l) < 0$ for $l \geq l_0$. Then, $(\mathcal{E})$ has a nonoscillatory solution.*

Proof

For any positive integer i, let $w^i(l)$ be a solution of $(\mathcal{E})$ such that

$$w^i(i) = 0, w^i(i+1) = 0, w^i(i+2) > 0.$$

Let $l \neq \theta_k$. Indeed, $w^i(l) > 0$ for $l \in \{0, 1, \dots, i-1\}$ due to

$$-c(l)w^i(l) = w^i(l+3) + a(l)w^i(l+2) + b(l)w^i(l+1).$$

Let $\{T^1(l), T^2(l), T^3(l)\}$ be a basis for the solution space of (1.1). For any solution $w^i(l)$ is therefore, we can write

$$w^i(l) = d_{1i}T^1(l) + d_{2i}T^2(l) + d_{3i}T^3(l)$$

with $d_{1i}^2 + d_{2i}^2 + d_{3i}^2 = 1$. Since the sequence $\{d_{ji}\}$, $j = 1, 2, 3$ is bounded, then there exists a subsequence i_k of $\{i\}$ such that $d_{ji_k} \to d_j$ as $k \to \infty$. Set

$$w(l) = d_1T^1(l) + d_2T^2(l) + d_3T^3(l).$$

Since $w^{i_k}(l) \to w(l)$ as $k \to \infty$, then it follows that $w(l) > 0$ for $l \geq l_0$. In this way, we find that $w(l)$ is a positive solution of (1.1).

Next, we consider the case when $l = \theta_k$. Because (1.1) leads to a similar mapping for $l = \theta_k$, without loss of generality, we assume impulsive perturbation at each step. Therefore, using (1.2) in (1.1), we have (3.12) noticing that $A(\theta_k) = \frac{\alpha_{k+3}a(\theta_k)}{\alpha_{k+2}}$, $B(\theta_k) = \frac{\alpha_{k+3}b(\theta_k)}{\alpha_{k+1}}$ and $C(\theta_k) = \frac{\alpha_{k+3}c(\theta_k)}{\alpha_k}$. Ultimately, $A(\theta_k) > 0$, $B(\theta_k) > 0$ and $C(\theta_k) < 0$ for $k \in \mathbb{N}$. For a positive integer l, let $v^i(\theta_k)$ be a solution of (3.12) such that

$$v^i(\theta_{k+i}) = 0, v^i(\theta_{k+i+1}) = 0, v^i(\theta_{k+i+2}) > 0.$$

Therefore, $v^i(\theta_{k+j}) > 0$ for $j \in \{0, 1, \dots, i-1\}$. Let $\{\mathcal{T}^1(l), \mathcal{T}^2(l), \mathcal{T}^3(l)\}$ be a basis of the solution space of (1.1) for which it is seldom to write

$$v^i(\theta_k) = e_{1i}\mathcal{T}^1(\theta_k) + e_{2i}\mathcal{T}^2(\theta_k) + e_{3i}\mathcal{T}^3(\theta_k)$$

with $e_{1i}^2 + e_{2i}^2 + e_{3i}^2 = 1$. Proceeding as above we can show that $w(\theta_k)$ is a positive solution of (3.12). Therefore, $w(l)$ is a positive solution (3.12) for all l and θ_k, $k \in \mathbb{N}$. This completes the proof of the theorem. $\square$

Discussion

In the first part of the main results (Theorem 3.1–3.6), we have used the method of contraction. Upon use of limiting behaviour of coefficients and impulsive inequalities, we are able to prove the main finding. For the proof of the next results (Theorem 3.9, 3.10), Lemma 3.8 plays a major role. First, we constructed two operators using appropriate solutions of $(\mathcal{E})$. By applying the properties of linear algebra, it was then possible to get oscillatory solutions of $(\mathcal{E})$. Although the method adopted here is simple, the research findings presented in this paper are also simpler and easily verifiable. Some numerical examples are presented to verify the obtained results.

Conclusion

The oscillation theory of third-order differential/difference equations has significantly advanced in the last few decades, as evidenced in the literature, see, e.g. [17–19, 23–25]. As we discussed in the introduction part, discrete-time systems are more computer-friendly than their continuous-time counterparts, so we have considered the third-order difference equations in a closed form rather than the usual discrete analogue. On the other hand, owing to many natural and man-made factors, the intrinsic growth of physical/biological processes usually undergoes some discrete changes of relatively short duration at fixed times. Often, such changes are characterised mathematically in the form of impulses [12–14, 26]. Therefore, in this work, we have studied a general class of third-order linear difference equations (1.1) under the influence of discrete moments of impulsive effects (1.2). Unlike most of the works in the literature, we do not require monotonic properties of nonoscillatory solutions to study $(\mathcal{E})$. In fact, this effort represents a new step in the literature of impulsive difference equations. For interested readers, this work can be extended to the nonlinear counterpart of $(\mathcal{E})$. We conclude this work by noticing that the system remains oscillating after the imposition of proper impulse controls.

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Declarations

Competing interests
The authors declare that they have no competing interests.

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