Ribeiro-Reis
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The kagebushin-beta distribution: an alternative for gamma, Weibull and exponentiated exponential distributions

Lucas David Ribeiro-Reis*

*Correspondence:
econ.lucasdavid@gmail.com

Department of Statistics, Federal University of Pernambuco, Recife, PE, Brazil

Abstract

A new lifetime distribution has been defined. This distribution is obtained from a transformation of a random variable with beta distribution and is called here the kagebushin-beta distribution. Some mathematical properties such as mode, quantile function, ordinary and incomplete moments, mean deviations over the mean and median and the entropies of Rényi and Shannon are demonstrated. The maximum likelihood method is used to obtain parameter estimates. Monte Carlo simulations are carried out to verify the accuracy of the maximum likelihood estimators. Applications to real data showed that the kagebushin-beta model can be better than the Weibull, gamma and exponentiated exponential distributions.

Keywords: Kagebushin-beta distribution, Rényi entropy, Shannon entropy, Mean deviations, Ordinary and incomplete moments

Mathematics Subject Classification: 60E05, 62F12, 65C05, 65C10

Introduction

A random variable Y having beta distribution has cumulative distribution function (cdf) and probability density function (pdf) given by

$$H(y;a,b)=\frac{1}{B(a,b)}\int_0^y z^{a-1}(1-z)^{b-1}\mathrm{dz},\quad \mathrm{y}\in(0,1) \tag{1}$$

and

$$h(y;a,b)=\frac{1}{B(a,b)}y^{a-1}(1-y)^{b-1},\quad y\in(0,1),$$

respectively, where $a>0$ and $b>0$ are shape parameters and $B(a,b)=\int_0^1 z^{a-1}(1-z)^{b-1}\mathrm{dz}$ denotes the beta function.

Taking the transformation $X=-\log Y$, the cdf and pdf of X are given by

$$F(x;a,b)=1-\frac{1}{B(a,b)}\int_0^{\mathrm{e}^{-x}} z^{a-1}(1-z)^{b-1}\mathrm{dz},\quad \mathrm{x}\in(0,\infty)$$

and

$$f(x; a, b) = \frac{1}{B(a, b)} \mathrm{e}^{-\mathrm{ax}} (1 - \mathrm{e}^{-\mathrm{x}})^{\mathrm{b}-1}, \quad \mathrm{x} \in (0, \infty), \tag{2}$$

respectively. Here, we refer to a as a scale parameter and b as a shape parameter. The random variable X with pdf (2) is said to have kagebushin-beta (KB) distribution and is denoted as $X \sim \mathrm{KB}(a, b)$.

The beta function admits the following relation

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \quad \text{and} \quad a\Gamma(a) = \Gamma(a + 1),$$

where $\Gamma(p) = \int_0^\infty z^{p-1} \mathrm{e}^{-\mathrm{z}} \mathrm{dz}$ denotes the gamma function. Under these results, for $b = 1$, the Equation (2) becomes

$$f(x; a, 1) = a\mathrm{e}^{-\mathrm{ax}}, \quad \mathrm{x} \in (0, \infty),$$

that corresponds to the pdf of the exponential distribution. Thus, the KB distribution has the exponential distribution as special case.

Figure 1 displays plots of the density function of X, for some values of the parameters.

This paper is organized as follows. In Sect. Properties, mathematical properties and entropy measures are described. In Sect. Estimation, the maximum likelihood method and Monte Carlo simulations are presented. In Sect. Applications, applications to real data are considered. Section Conclusions concludes the paper.

Properties

The first derivative of the log-density (2) is

$$\eta(x) = \frac{\mathrm{d}}{\mathrm{dx}} \log f(x; a, b) = -a + \frac{(b - 1)\mathrm{e}^{-\mathrm{x}}}{1 - \mathrm{e}^{-\mathrm{x}}}.$$

The mode is obtained by solution of $\eta(x) = 0$. So, the mode of X is

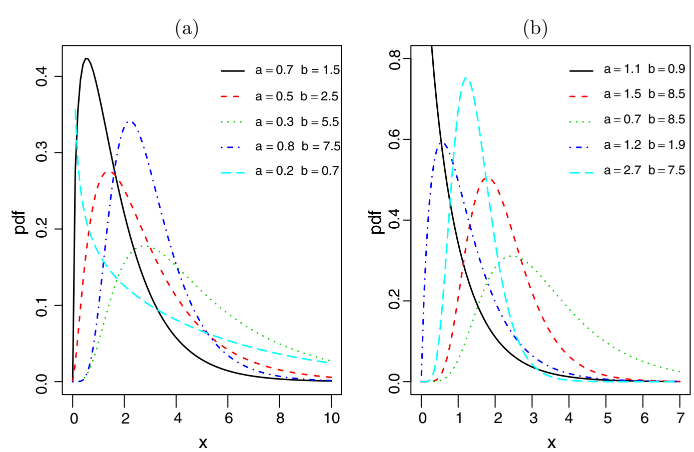

Fig. 1 Some pdfs of the KB distribution

$$\mathrm{mode}(X) = \begin{cases} 0, & b \leq 1, \\ -\log\left(\frac{a}{a+b-1}\right), & \text{otherwise.} \end{cases}$$

By inverting $F(x; a, b) = u$, the quantile function of X is

$$F^{-1}(u; a, b) = -\log Q_1(1-u; a, b), \quad u \in (0,1),$$

where $Q_1(\cdot; a, b)$ is the inverse function of the Equation (1). Using the quantile function, the random variable

$$X = -\log Q_1(1-V; a, b) \quad \text{or} \quad X = -\log Q_1(V; a, b) \tag{3}$$

has density function (2), where V is a uniform random variable over the interval (0, 1).

The rth moment of X is obtained as

$$\mu^r = \mathbb{E}[X^r] = \frac{1}{B(a,b)} \int_0^\infty x^r \mathrm{e}^{-\mathrm{ax}} (1-\mathrm{e}^{-\mathrm{x}})^{\mathrm{b}-1} \mathrm{dx}.$$

Consider the following convergent expansion in power series

$$(1-\mathrm{e}^{-\mathrm{x}})^{\mathrm{b}-1} = \sum_{\mathrm{k}=0}^{\infty} (-1)^{\mathrm{k}} \binom{b-1}{k} \mathrm{e}^{-\mathrm{kx}}.$$

Using the expansion above, the rth moment of X can be written as

$$\mu^r = \frac{1}{B(a,b)} \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} \int_0^\infty x^r \mathrm{e}^{-(\mathrm{a}+\mathrm{k})\mathrm{x}} \mathrm{dx}.$$

Taking $w = (a+k)x$, we have

$$\begin{aligned} I &= \int_0^\infty x^r \mathrm{e}^{-(\mathrm{a}+\mathrm{k})\mathrm{x}} \mathrm{dx} \\ &= \frac{1}{(a+k)^{r+1}} \int_0^\infty w^r \mathrm{e}^{-\mathrm{w}} \mathrm{dw} \\ &= \frac{\Gamma(r+1)}{(a+k)^{r+1}}. \end{aligned}$$

So, the rth moment of X is given by

$$\mu^r = \frac{1}{B(a,b)} \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} \frac{\Gamma(r+1)}{(a+k)^{r+1}}.$$

For $s > 0$, the rth incomplete moment of X is obtained as

$$\begin{aligned} m_r(s) &= \frac{1}{B(a,b)} \int_0^s x^r \mathrm{e}^{-\mathrm{ax}} (1-\mathrm{e}^{-\mathrm{x}})^{\mathrm{b}-1} \mathrm{dx} \\ &= \frac{1}{B(a,b)} \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} \int_0^s x^r \mathrm{e}^{-(\mathrm{a}+\mathrm{k})\mathrm{x}} \mathrm{dx}. \end{aligned}$$

Taking $t = (a+k)x$, we have

$$\begin{aligned} J &= \int_0^s x^r \mathrm{e}^{-(\mathrm{a+k})\mathrm{x}} \mathrm{dx} \\ &= \frac{1}{(a+k)^{r+1}} \int_0^{(a+k)s} t^r \mathrm{e}^{-\mathrm{t}} \mathrm{dt} \\ &= \frac{\gamma(r+1,(a+k)s)}{(a+k)^{r+1}}, \end{aligned}$$

where $\gamma(p,x) = \int_0^x z^{p-1}\mathrm{e}^{-\mathrm{z}}\mathrm{dz}$ denotes the lower incomplete gamma function.

Then, the rth incomplete moment of X is given by

$$m_r(s) = \frac{1}{B(a,b)} \sum_{k=0}^{\infty} (-1)^k \binom{b-1}{k} \frac{\gamma(r+1,(a+k)s)}{(a+k)^{r+1}}. \tag{4}$$

An entropy is a measure of variation or uncertainty of a random variable. Two popular entropy measures are the Rényi and Shannon entropies. For $\rho > 0$ and $\rho \neq 1$, the Rényi entropy of a random variable having pdf $f(\cdot)$ with support in (a, b) is given by

$$\mathcal{I}_R(\rho) = \frac{1}{1-\rho} \log\left(\int_a^b f(x)^\rho \mathrm{dx}\right).$$

For KB distribution, the Rényi entropy is

$$\mathcal{I}_R(\rho) = \frac{1}{1-\rho} \log\left(\frac{1}{B(a,b)^\rho} \int_0^\infty \mathrm{e}^{-\mathrm{a}\rho\mathrm{x}} (1-\mathrm{e}^{-\mathrm{x}})^{(\mathrm{b}-1)\rho} \mathrm{dx}\right).$$

Setting $v = \mathrm{e}^{-\mathrm{x}}$, we have

$$\begin{aligned} L &= \int_0^\infty \mathrm{e}^{-\mathrm{a}\rho\mathrm{x}} (1-\mathrm{e}^{-\mathrm{x}})^{(\mathrm{b}-1)\rho} \mathrm{dx} \\ &= \int_0^1 v^{a\rho-1} (1-v)^{(b-1)\rho} dv \\ &= B(a\rho, (b-1)\rho+1). \end{aligned}$$

Thus, the Rényi entropy of X becomes

$$\mathcal{I}_R(\rho) = \frac{1}{1-\rho} \log\left(\frac{B(a\rho,(b-1)\rho+1)}{B(a,b)^\rho}\right).$$

The Shannon entropy is given by $\mathcal{I}_s = \mathbb{E}[-\log f(X)]$. So, for KB distribution, the Shannon entropy is

$$\mathcal{I}_S = \log B(a,b) + a\mathbb{E}[X] - (b-1)\mathbb{E}[\log(1-\mathrm{e}^{-\mathrm{X}})].$$

From the maximum likelihood method, we can show that $\mathbb{E}[X] = \psi(a+b) - \psi(a)$ and $\mathbb{E}[\log(1-\mathrm{e}^{-\mathrm{X}})] = \psi(\mathrm{b}) - \psi(\mathrm{a+b})$, where $\psi(p) = d\log\Gamma(p)/dp$ is the digamma function.

Thus, the Shannon entropy is of X is

$$\mathcal{I}_S = \log B(a,b) + a\psi(a+b) - a\psi(a) - (b-1)[\psi(b) - \psi(a+b)].$$

Thus, we see that for KB distribution, the Rényi and Shannon entropies can be easily computed.

The mean deviations of X about the mean and about the median are given as

$$\begin{aligned}\varphi_1(\mu^1) &= \int_0^1 |x-\mu^1| f(x;a,b)\mathrm{dx}\\ &= 2\mu^1 F(\mu^1,a,b) - 2m_1(\mu^1)\end{aligned}$$

and

$$\begin{aligned}\varphi_2(\omega) &= \int_0^1 |x-\omega| f(x;a,b)\mathrm{dx},\\ &= \mu^1 - 2m_1(\omega),\end{aligned}$$

respectively, where $\mu^1 = \mathbb{E}[X]$ and $\omega = F^{-1}(0.5;a,b)$ and $m_1(\cdot)$ is defined in (4).

Estimation

Let the random variables $X_i,\cdots,X_n \sim \mathrm{KB}(a,b)$ with observed values $x_i,\cdots,x_n$. From Equation (2), the log-likelihood for $(a,b)^\top$ is given by

$$\mathcal{L}(a,b) = -n\log B(a,b) - a\sum_{i=1}^{n} x_i + (b-1)\sum_{i=1}^{n}\log(1-\mathrm{e}^{-\mathrm{x_i}}).$$

The components of the score vector $U(a,b) = (U_a,U_b)^\top$ of $\mathcal{L}(a,b)$ are given by

$$\begin{aligned}U_a &= n\psi(a+b) - n\psi(a) - \sum_{i=1}^{n} x_i,\\ U_b &= n\psi(a+b) - n\psi(b) + \sum_{i=1}^{n}\log(1-\mathrm{e}^{-\mathrm{x_i}}).\end{aligned}$$

The maximum likelihood estimates (MLEs) of a and b, say $\hat{a}$ and $\hat{b}$, are the simultaneous solutions of $U_a = U_b = 0$, which has no closed forms. Thus, this problem can be solved via iteractive numerical methods, such that Newton--Raphson algorithmic. Statistical packages such as `R` [1] and `Ox` [2] can be used for this purpose.

The Fisher expected information matrix is

$$\mathcal{K}(a,b) = -\begin{bmatrix} U_{aa} & U_{ab}\\ U_{ba} & U_{bb}\end{bmatrix},$$

in which

$$\begin{aligned}U_{aa} &= n\psi'(a+b) - n\psi'(a),\\ U_{ab} &= U_{ba} = n\psi'(a+b),\\ U_{bb} &= n\psi'(a+b) - n\psi'(b),\end{aligned}$$

where $\psi'(p) = d^2\log\Gamma(p)/dp^2$ is the trigamma function.

Under general regularity conditions, we have the result

Table 1 Monte Carlo simulation results for scenario 1

Par	$n = 25$			$n = 50$		
	AE	**Bias**	**MSE**	**AE**	**Bias**	**MSE**
a	2.13855	0.23855	0.47691	2.01349	0.11349	0.17800
b	1.67733	0.17733	0.27864	1.58523	0.08524	0.10471
Par	$n = 75$			$n = 100$		
	AE	Bias	MSE	AE	Bias	MSE
a	1.97501	0.07501	0.11092	1.95472	0.05472	0.07862
b	1.55656	0.05656	0.06435	1.54096	0.04097	0.04573
Par	$n = 200$			$n = 400$		
	AE	Bias	MSE	AE	Bias	MSE
a	1.92778	0.02778	0.03657	1.91222	0.01222	0.01752
b	1.52129	0.02129	0.02155	1.50967	0.00967	0.01013

Table 2 Monte Carlo simulation results for scenario 2

Par	$n = 25$			$n = 50$		
	AE	**Bias**	**MSE**	**AE**	**Bias**	**MSE**
a	5.09265	0.59265	2.85719	4.78259	0.28259	1.06584
b	2.80695	0.30695	0.80707	2.64745	0.14745	0.30248
Par	$n = 75$			$n = 100$		
	AE	Bias	MSE	AE	Bias	MSE
a	4.68684	0.18684	0.66211	4.63623	0.13623	0.46898
b	2.59784	0.09785	0.18606	2.57095	0.07095	0.13207
Par	$n = 200$			$n = 400$		
	AE	Bias	MSE	AE	Bias	MSE
a	4.56945	0.06945	0.21822	4.53067	0.03067	0.10410
b	2.53680	0.03680	0.06208	2.51663	0.01663	0.02919

$$((\hat{a}, \hat{b}) - (a, b)) \quad \overset{a}{\sim} \quad \mathcal{N}_2(0, \mathcal{K}(a, b)^{-1}),$$

where $\mathcal{K}(a, b)^{-1}$ is the inverse matrix of $\mathcal{K}(a, b)$ and $\overset{a}{\sim}$ denotes asymptotic distribution. This multivariate normal approximation for $(\hat{a}, \hat{b})$ can be used for construing approximate confidence intervals for the model parameters. The LR statistics can be used for testing hypotheses on these parameters.

Simulation study

To show the accuracy of MLEs for the two parameters of the KB model, Monte Carlo simulations with $15,000$ replications were performed. Two scenarios are considered and the sample sizes chosen are $n = \{25, 50, 75, 100, 200, 400\}$. The random numbers are generated using Equation (3). The true parameters are: $a = 1.9$ and $b = 1.5$ in scenario 1 and $a = 4.5$ and $b = 2.5$ in scenario 2. The simulations were carried out using the matrix programming language `Ox` [2].

Tables 1 and 2 list the average estimates (AEs), biases and mean squared errors (MSEs), for scenarios 1 and 2, respectively. As expected, the MLEs converge to the true parameters and the biases and MSEs decrease when the sample size n increases.

Applications

In this section, we compare the results of fitting of the KB distribution with three others well-known distributions, for two datasets.

The data are:

- (Dataset 1) The data refer to remission times (in months) of 128 bladder cancer patients. These data were also analyzed by [3].
- (Dataset 2) The data consist of the waiting time between 64 consecutive eruptions of the Kiama Blowhole [4].

We compare the KB model (2) with the Weibull, gamma and exponentiated exponential [5] distributions. The pdfs of the Weibull (W), gamma (G) and exponentiated exponential (EE) distributions are

$$f_{\mathrm{W}}(x;\lambda,\beta)=\beta\lambda^{\beta}x^{\beta-1}\,\mathrm{e}^{-(\lambda \mathrm{x})^{\beta}},\quad \mathrm{x}>0,$$
$$f_{\mathrm{g}}(x;\lambda,\beta)=\frac{\lambda^{\beta}}{\Gamma(\beta)}x^{\beta-1}\,\mathrm{e}^{-\lambda \mathrm{x}},\quad \mathrm{x}>0$$

and

$$f_{\mathrm{ee}}(x;\lambda,\beta)=\beta\lambda\mathrm{e}^{-\lambda\mathrm{x}}(1-\mathrm{e}^{-\lambda\mathrm{x}})^{\beta-1},\quad \mathrm{x}>0,$$

respectively, where $\lambda>0$ is scale parameter and $\beta>0$ is shape parameter. Note that for $\beta=1$, all these pdfs become the density function of the exponential distribution.

The goodness-of-fit measures adopted are: Cramér-von Mises (W^*), Anderson Darling (A^*), Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC) for model comparisons. The lower the value of these statistics, more evidence we have for a good fit. The graphical analysis is also important to identify the best fitted model. All the computations were done using the `Ox` language [2].

Tables 3 and 5 list the MLEs with standard errors in parentheses (SEs) for datasets 1 and 2, respectively. Note that, in both applications, all MLEs of all models are significant, since their standard errors are low, when compared to the respective MLE.

Table 3 MLEs and SEs for dataset 1

Model	Estimates	
KB(a, b)	0.1138	1.5442
	(0.0106)	(0.3307)
W(λ, β)	0.1046	1.0478
	(0.0093)	(0.0676)
G(λ, β)	0.1252	1.1725
	(0.0173)	(0.1308)
EE(λ, β)	0.1212	1.2180
	(0.0136)	(0.1488)

The information criteria for datasets 1 and 2 are presented in Tables 4 and 6 , respectively. In both datasets, all information criteria point to the KB distribution as the best model, followed later by the EE distribution.

Figures 2 and 3 show the estimated pdfs and cdfs for datasets 1 and 2, respectively, considering the two best fitted models.

Conclusions

A new lifetime distribution has been defined. This distribution is obtained from a transformation of a random variable with beta distribution and is called here the kagebushin-beta distribution. Some mathematical properties such as mode, quantile function, ordinary and incomplete moments, mean deviations over the mean and median and the entropies of Rényi and Shannon are demonstrated.

The method used to estimate the parameters was maximum likelihood. Fisher's expected information matrix has closed form. Monte Carlo simulations showed that the maximum likelihood estimators of the new model are valid, being in accordance with the asymptotic theory.

The usefulness of the kagebushin-beta model is shown with applications to real data. The results of these applications showed that the kagebushin-beta model is better than the Weibull, gamma and exponentiated exponential distributions.

Table 4 Information criteria for dataset 1

Model	W^*	A^*	CAIC	AIC	BIC	HQIC
KB	0.0847	0.5939	829.0001	828.9041	834.6082	831.2217
W	0.1148	0.8636	832.2698	832.1738	837.8778	834.4913
G	0.1050	0.7902	830.8316	830.7356	836.4396	833.0531
EE	0.0985	0.7409	830.2512	830.1552	835.8592	832.4728

Table 5 MLEs and SEs for dataset 2

Model	Estimates	
KB(a, b)	0.0304	637.4722
	(0.0038)	(20.9722)
W(λ, β)	0.0231	1.2745
	(0.0024)	(0.1203)
G(λ, β)	0.0407	1.6208
	(0.0077)	(0.2623)
EE(λ, β)	0.0350	1.7315
	(0.0051)	(0.3199)

Table 6 Information criteria for dataset 2

Model	W^*	A^*	CAIC	AIC	BIC	HQIC
KB	0.0774	0.4566	582.9394	582.7427	587.0605	584.4437
W	0.1310	1.0864	597.9970	597.8003	602.1180	599.5012
G	0.1220	0.9850	595.9955	595.7988	600.1166	597.4998
EE	0.1206	0.9580	595.5288	595.3320	599.6498	597.0330

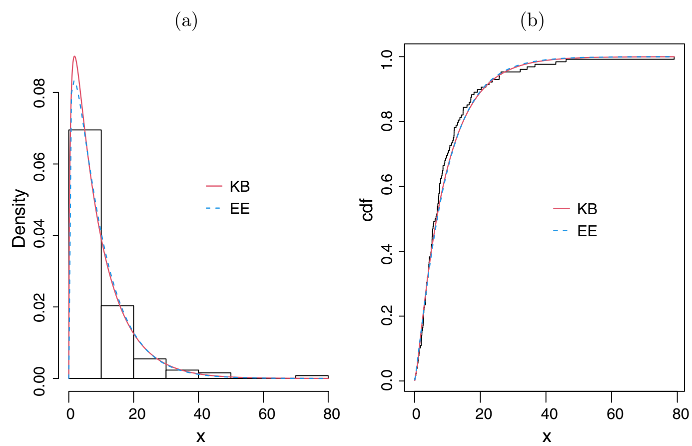

Fig. 2 Estimated **a** pdfs and **b** cdfs for data 1

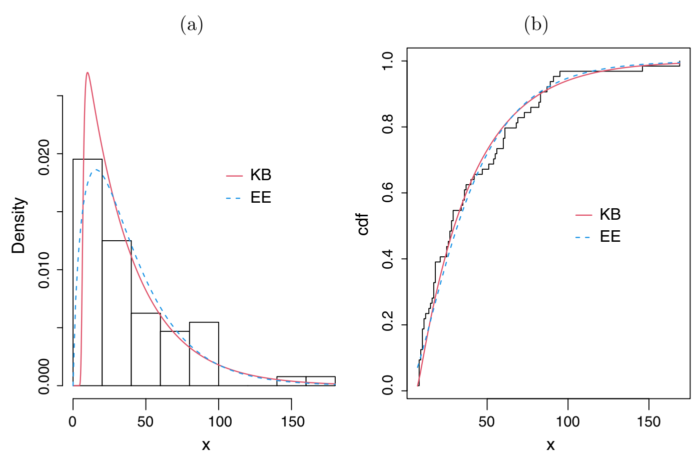

Fig. 3 Estimated **a** pdfs and **b** cdfs for data 2

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