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A convenient category of topological partial groups

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Abstract
In this paper, the concept of $\wp$-continuous map is introduced and some of their basic properties are discussed. Also, the category $\acute{\mathbf{K}}$, of topological partial groups, as objects and the $\wp$-morphisms of topological partial groups, as arrows, is introduced, which is alternative to the category **K**, of topological spaces, as objects and k-continuous maps, as arrows, and satisfies the same nice properties of the category **kpg**, of $\underline{k}$-partial groups, as objects, and the morphisms of $\underline{k}$-partial groups, as arrows (Abd- Allah et al., J. Egyption Math. Soc 25:276-278, 2017).

Keywords: Partial group, Partial group homomorphism, Topological group, Topological partial group, $\underline{k}$-partial group

MSC: 22A05, 22A10, 22A20, 54H11

Introduction

In [1], A.M. Abd- Allah et al. introduced the concept of topological partial groups and discussed some of their basic properties. Also, they introduced the category **Tpg** of topological partial groups, as objects and the homorphisms of topological partial groups, as arrows. So, the category **Tpg** has the following deficiencies:

(i) If $a \in S$, then the right transformation $r_a : S \to S, x \mapsto xa$ and the left transformation $l_a : S \to S, x \mapsto ax$, may not be open.

(ii) The quotient map $\rho_N : S \to S, x \mapsto xN, N \leq S$, may not be open, in general, where S/N has the identification topology with respect to the quotient map.

(iii) Let S be a topological partial group and $N \trianglelefteq S$. Then, the partial group S/N may not be a topological partial group, since the cartesian product of two identification maps may not be identification.

In [2], A.M. Abd- Allah et al. introduced the concept of $\underline{k}$-partial groups and discussed some of their basic properties. Also, they introduced the category **kpg**, of $\underline{k}$-partial groups, as objects, and the morphisms of $\underline{k}$-partial groups, as arrows which is modified the above deficiencies. In this paper, the concept of $\wp$-continuous maps is introduced and some of their basic properties are discussed. Also, the category $\acute{\mathbf{K}}$ of topological partial groups, as objects, and the $\wp$-morphisms of topological partial groups , as arrows, is introduced, which is alternative to the category **K**, of topological spaces, as objects, and k-continuous maps, as arrows. The category $\acute{\mathbf{K}}$ satisfied the same nice properties of the category **kpg**. The idea of $\wp$-continuous maps was taken from the definition of k-continuous map [3].

Preliminaries

We collect for sake of reference the needed definitions and results appeared in the given references.

Definition 1 [4] *Let S be a semigroup. Then, $x \in S$ is called an idempotent element if $x \cdot x = x$. The set of all idempotent elements in S is denoted by $E(S)$.*

Definition 2 [5] *Let S be a semigroup and $x \in S$. Then, an element $e \in S$ is called a partial identity of x if:*

(i) $ex = xe = x$,
(ii) *If $e'x = xe' = x$, for some $e' \in S$, then $ee' = e'e = e$.*

Theorem 1 [5] *Let S be a semigroup. Then,*

(i) *If $x \in S$ has a partial identity, then it is unique*
(ii) *$E(S)$ is the set of all partial identities of the elements of S.*

We will denote by e_x the partial identity of the element $x \in S$.

Definition 3 [5] *Let S be a semigroup and $x \in S$ has a partial identity element e_x. Then, $y \in S$ is called a partial inverse of x if:*

(i) $xy = yx = e_x$,
(ii) $e_x y = ye_x = y$.

We will denote the partial inverse y of $x \in S$ by x^{-1} .

Definition 4 [5] *A semigroup S is called a partial group if:*

(i) *Every $x \in S$ has a partial identity e_x*
(ii) *Every $x \in S$ has a partial inverse x^{-1}*
(iii) *The map $e_S : S \to S, x \mapsto e_x$ is a semigroup homomorphism*
(iv) *The map $\gamma : S \to S, x \mapsto x^{-1}$ is a semigroup antihomomorphism.*

So, every group is a partial group.

Definition 5 [6] *Let S be a partial group and $x \in S$. Then, we define $S_x = \{y \in S : e_x = e_y\}$.*

Theorem 2 [5] *Let S be a partial group and $x \in S$. Then,*

(i) *S_x is a maximal subgroup of S which has identity e_x*
(ii) $S = \bigcup\{S_x : x \in S\}$.

Corollary 1 [5] *Every partial group is a disjoint union of a family of groups.*

Definition 6 [3] *Let X be a topological space. Then, the map $\alpha : C \to X$ is called a test map if α is continuous and C is a compact Hausdorff space.*

Definition 7 [3] *Let X and Y be topological spaces. Then, the map $f : X \to Y$ is called k-continuous if $f\alpha : C \to Y$ is continuous, for each test map $\alpha : C \to X$.*

Let $\boldsymbol{\tau}$ be the category of topological spaces, as objects and continuous maps, as arrows. Also, let $\mathbf{K}$ be the category of topological spaces, as objects and k-continuous maps, as arrows. It is clear that the category $\boldsymbol{\tau}$ is a wide subcategory of $\mathbf{K}$.

Definition 8 [1] *Let S be a partial group and τ be a topology on S. Then, S is called a topological partial group if the following maps are continuous:*

(i) *The product map: $\mu : S \times S \to S, (x, y) \mapsto xy$*
(ii) *The partial identity map: $e_S : S \to S, x \mapsto e_x$*
(iii) *The partial inverse map: $\gamma : S \to S, x \mapsto x^{-1}$.*

Definition 9 [1] *Let S be a topological partial group and $a \in S$. Then, the map $r_a : S \to S, x \mapsto xa$ is called a right transformation and the map $l_a : S \to S, x \mapsto ax$ is called a left transformation.*

Theorem 3 [1] *The maps r_a and l_a are continuous.*

Let $\boldsymbol{\wp}$ be a non-empty full subcategory of $\boldsymbol{\tau}$ which satisfies the following conditions [7]:

(i) If A is a closed subspace of an object B of $\boldsymbol{\wp}$, then A is a $k\wp$-space.
(ii) If B and C are objects in $\boldsymbol{\wp}$, then $B \times C$ is also object in $\boldsymbol{\wp}$.
(iii) For objects X in $\boldsymbol{\wp}$ and Y in $\boldsymbol{\tau}$, the evaluation map $e : Y^X \times X \to Y, (f, x) \mapsto f(x)$ and $x \in X$, is continuous, where Y^X has the compact open topology.
(iv) If A and B are objects in $\boldsymbol{\wp}$, then the topological sum $A \bigsqcup B$ is also an object in $\boldsymbol{\wp}$.

Definition 10 [2] *Let S be a topological partial group. Then, the map $h : C \to S$ is called a $\wp$-test map if h is continuous and $h^{-1}(S_{e_x})$ is open in C for each $e_x \in E(S)$, where $C \in obj(\wp)$.*

$\wp$-continuous maps

In this section, the notion of $\wp$-continuous map is introduced and some of their basic properties are discussed. Also, the category $\mathbf{K}^{\wp}$ of $\wp$-continuous maps, as objects and the morphisms of $\wp$-continuous maps as arrows, is introduced.

Definition 11 *Let S and T be topological partial groups. Then, the map $f : S \to T$ is called $\wp$-continuous if $fh : C \to T$ is continuous, for each a $\wp$-test map $h : C \to S$.*

We note that every continuous map of topological partial group is $\wp$-continuous. So, the following maps are $\wp$-continuous:

(i) The identity map $I : S \to S$
(ii) The partial identity map: $e_S : S \to S, x \mapsto e_x$
(iii) The partial inverse map: $\gamma : S \to S, x \mapsto x^{-1}$
(iv) The maps r_a and l_a.

Definition 12 *Let $f : S \to T$ be a $\wp$-continuous map. Then, f is called a $\wp-$morphism if it is a partial group homomorphism.*

We note that (i) and (iii) above are $\wp$-morphisms.

Theorem 4 *If $f : S \to T$ and $g : T \to F$ are $\wp$-morphisms, then $gf : S \to F$ is also a $\wp$-morphism.*

Proof It is clear that gf is a partial group homomorphism. Let $h : C \to S$ be a $\wp-$test map. Since f is $\wp$-continuous, then $fh : C \to T$ is continuous. Now, $(fh)^{-1}(T_e) = h^{-1}(f^{-1}(T_e))$, for each $e \in E(T)$. Since f is a partial group homomorphism, then $f^{-1}(T_e)$ is a maximal subgroup of S. So, $(fh)^{-1}(T_e)$ is open in C, for each $e \in E(T)$. That means that fh is a $\wp$-test map. Since g is $\wp$-continuous, then $g(fh) = (gf)h : C \to F$ is continuous. Then, gf is $\wp$-continuous. Hence, gf is a $\wp$-morphism. □

Definition 13 *A subset V of the topological partial group S is called $\wp$-open if $h^{-1}[V]$ is open in C for each a $\wp$-test map $h : C \to S$*

From the above definition, we have that S_{e_x} is $\wp$-open in S.

Theorem 5 *The family $\{\wp - \tau_S\}$ of $\wp$-open sets form a topology on S.*

Proof It is clear that ϕ and S are $\wp$-open sets, since $h^{-1}[S] = C$ and $h^{-1}[\phi] = \phi$. If U and V are $\wp$-open sets, then $h^{-1}[U]$ and $h^{-1}[V]$ are open sets in C. But $h^{-1}[U \bigcap V] = h^{-1}[U] \bigcap h^{-1}[V]$ is open in C. So, $U \bigcap V$ is a $\wp$-open set. Similarly, let $(U_\lambda)_{\lambda \in L}$ be a sub-family of $\wp$-open sets. Then, $h^{-1}[U_\lambda]$ are open in C, for each $\lambda \in L$. Since $h^{-1}[\bigcup_\lambda U_\lambda] = \bigcup_\lambda h^{-1}[U_\lambda]$ is open in C. Hence, $\bigcup U_\lambda$ is a $\wp$-open set. □

Definition 14 *A subset A of the topological partial group S is called a $\wp$-neighbourhood of $x \in S$ if there exists a $\wp$-open set U in S such that $x \in U \subseteq A$.*

The family of all $\wp$-neighbourhoods of $x \in S$ is called a $\wp$-neighbourhood system and is denoted by $\wp - N_x$

Proposition 1 *A subset $A \subseteq S$ of the topological partial group S is a $\wp$-open set if and only if it is a $\wp$-neighbourhood of each of its points.*

Proof Let A be a $\wp$-open set. Then, $x \in A \subseteq A$, for all $x \in A$. Hence, A is a $\wp$-neighbourhood of x. Conversely, for each $x \in A$, there exists a $\wp$-open set U_x such that $x \in U_x \subseteq A$. So, $A = \bigcup_{x \in A} U_x$. Hence, A is a $\wp$-open set. □

Theorem 6 *Let S be a topological partial group and $x \in S$. Then,*

(i) $x \in N$, *for all* $N \in N_x$
(ii) *If* $N \in N_x$ and $N \subseteq M$, *then* $M \in N_x$
(iii) *If* $N, M \in N_x$, *then* $N \bigcap M \in N_x$
(iv) *If* $N \in N_x$, *then there exists* $M \in N_x$ *such that* $N \in N_y$, *for each* $y \in M$.

Proof (i) If $N \in N_x$, then there exists a $\wp$-open set U in S such that $x \in U \subseteq N$. Hence, $x \in N$.
(ii) If $N \in N_x$, then there exists a $\wp$-open set U in S such that $x \in U \subseteq N$. Since, $N \subseteq M$, then $x \in U \subseteq M$. Hence, $M \in N_x$.
(iii) If $N, M \in N_x$, then there exist two $\wp$-open sets U and V, respectively such that $x \in U \subseteq N$ and $x \in V \subseteq M$. So, we have that $x \in U \bigcap V \subseteq N \bigcap M$. Since $N \bigcap M$ is a $\wp$-open set, then $N \bigcap M \in N_x$.

(iv) If $N \in N_x$, then there exists a $\wp$-open set M in S such that $x \in M \subseteq N$. Since M is a $\wp$-open set, then $M \in N_y$, for all $y \in M$. Since, $N \subseteq M$, then $N \in N_y$, for each $y \in M$. □

Definition 15 *Let S be a topological partial group and $A \subseteq S$. Then, $x \in A$ is called a $\wp$-interior point of A if A is a $\wp$-neighbourhood of x.*

The set of all $\wp$-interior points of A is called $\wp$-interior set and is denoted by $\wp - A^0$.

Proposition 2 *Let S be a topological partial group and $A, B \subseteq S$. Then,*

(i) $\wp - A^0 \subseteq A$
(ii) *If $A \subseteq B$, then $\wp - A^0 \subseteq \wp - B^0$*
(iii) $\wp - A^0$ *is a $\wp$-open set*
(iv) $(\wp - A^0)^0 = \wp - A^0$.

Proof (i) Let $x \in \wp - A^0$. Then, $A \in N_x$. So, $x \in A$.
(ii) Let $x \in \wp - A^0$. Then, $A \in N_x$. Since, $A \subseteq B$, then $B \in N_x$ and so $x \in \wp - B^0$. Hence, $\wp - A^0 \subseteq \wp - B^0$.
(iii) Let $x \in \wp - A^0$. Then, $A \in N_x$. Thus, there exists $N \in N_x$ such that $A \in N_y$, for all $y \in N$. That is, $y \in \wp - A^0$, for all $y \in N$. Hence, $N \subseteq A$. Thus, $x \in N \subseteq \wp - A^0$. So, $A \in N_x$. Therefore, $\wp - A^0$ is a $\wp$-open set.
(iv) Since $\wp - A^0 \subseteq A$, then from (ii) $(\wp - A^0)^0 \subseteq \wp - A^0$. It remains that $\wp - A^0 \subseteq (\wp - A^0)^0$. This is given from $x \in \wp - A^0$. That is, $\wp - A^0 \in N_x$. Hence, $x \in (\wp - A^0)^0$. □

Corollary 2 *A subset A of the topological partial group S is $\wp$-open if and only if $\wp - A^0 = A$.*

Proof It is obvious. □

Definition 16 *A subset A of the topological partial group S is called $\wp$-closed if $S - A$ is a $\wp$-open set.*

Definition 17 *Let S be a topological partial group and $A \subseteq S$. Then, $x \in S \in$ is called a $\wp$-closure point of A if $A \bigcap N \neq \phi$, for each $N \in \wp - N_x$.*

The set of all $\wp$-closure points of A is called the $\wp$-closure of A and is written by $\wp - \overline{A}$.

Proposition 3 *Let A be a subset of the topological partial group S. Then, the family $\tau_A = \{U \bigcap A : U \text{ is } \wp - \text{open in } S\}$ is a topology on A, which is called $\wp$-relative topology.*

Proof It is clear that $\phi, A \in \tau_A$ since $\phi = \phi \bigcap A$ and $A = A \bigcap S$. Let $M, N \in \tau_A$. Then, there exist two $\wp$-open sets U and V such that $M = U \bigcap A$ and $N = V \bigcap A$. So, $M \bigcap N \in \tau_A$. Also, let $V = (V_\lambda)_{\lambda \in L}$ be a subfamily of τ_A. Then, for each λ, there are $\wp$-open sets U_λ such that $V = U_\lambda \bigcap A$. Then, $V = \bigcup_{\lambda \in L} V_\lambda = \bigcup_{\lambda \in L}(U_\lambda \bigcap A) = (\bigcup_{\lambda \in L} U_\lambda) \bigcap A$. □

Theorem 7 *Let $f : S \to T$ be $\wp$-continuous. Then, $f \mid A : A \to T$ is $\wp$-continuous.*

Proof Let $U \subseteq T$ be $\wp$-open. Now, $(f \mid A)^{-1}(U) = f^{-1}(U) \bigcap A$. Since $f^{-1}(U)$ is a $\wp-$open set in S, then $f^{-1}(U)$ is a $\wp$-open in A. □

Definition 18 *Let S be a topological partial group and A be a subpartial group of S. Then, A with the $\wp$-relative topology is a topological partial group, called a topological subpartial group, denoted by $A \leq S$.*

Definition 19 *Let S and T be topological partial groups and let $(x,y) \in S \times T$. The set $\wp - (S \times T)$, where $M \in N_x$ in S and $N \in N_y$ in T is called a $\wp$-basic neighbourhood of (x,y).*

Definition 20 *A subset U of $M \times N$ is called a $\wp$-neighbourhood if there exists a $\wp$-basic neighbourhood $M \times N$ of (x,y) such that $(x,y) \in M \times N \subseteq U$.*

We note that if M and N are $\wp$-open sets in the topological partial groups S and T, respectively, then $M \times N$ is a $\wp$-basic neighbourhood of any $(x,y) \in M \times N$.

Theorem 8 *(i) If A and B are $\wp$-open sets in S and T, respectively, then $A \times B$ is also $\wp$-open in $S \times T$*

(ii) If C and D are $\wp$-closed sets in S and T, respectively, then $C \times D$ is also $\wp$-closed in $S \times T$.

Proof (i) Let $(x,y) \in U \times V$. Then, $x \in U$ and $y \in V$. So, $U \in \wp - N_x$ in S and $V \in \wp - N_y$ in T. This implies $U \times V$ is a $\wp$-basic neighbourhood of (x,y). Since $(x,y) \in U \times V \subseteq A \times B$, then $U \times V \in N_{(x,y)}$. Hence, $A \times B$ is also $\wp$-open in $S \times T$.

(ii) We have $(S \times T) - (C \times D) = (S - C) \times T \bigcup S \times (T - D)$. Since $S - C$ and $T - D$ are $\wp$-open sets in S and T, respectively, then $(S - C) \times T$ and $S \times (T - D)$ are $\wp$-open sets in $S \times T$ and so $(S \times T) - (C \times D)$ is $\wp$-open set in $S \times T$. That is, $C \times D$ is $\wp$-closed in $S \times T$.

□

We note that the following maps are $\wp$-continuous, for each topological partial group S:

(i) The projection maps $P_1 : S \times T \to S$ and $P_2 : S \times T \to T$.
(ii) The product map $\mu : S \times S \to S$.
(iii) The diagonal map $\Delta_S = \{(x,x) : x \in S\}$.

Theorem 9 *If $f : S \to T$ and $f : S \to F$ are $\wp$-morphisms, then $(f,g) : S \to T \times F$ is also a $\wp$-morphism.*

Proof It is clear that (f,g) is a partial group homomorphism. Let $h : C \to S$ be a $\wp$-test map. Since f is $\wp$-continuous, then $fh : C \to T$ is continuous. Also, since g is $\wp$-continuous, then $gh : C \to T$ is continuous. So, $(fh, gh) = (f,g)h : S \to T \times F$ is continuous. That is, (f,g) is $\wp$-continuous. Hence, (f,g) is a $\wp$-morphism. □

Theorem 10 *If $f_1 : S_1 \to T_1$ and $f_2 : S_2 \to T_2$ are $\wp$-morphisms, then $f_1 \times f_2 : S_1 \times S_2 \to T_1 \times T_2$ is also a $\wp$-morphism.*

Proof It is clear that $f_1 \times f_2 : S_1 \times S_2 \to T_1 \times T_2$ is a partial group homomorphism. Since $f_1 \times f_2 = (f_1 P_1, f_2 P_2)$, then from the last theorem, we have that $f_1 \times f_2$ is $\wp$-continuous. Hence, $f_1 \times f_2$ is a $\wp$-morphism. □

Theorem 11 *Let S and T be topological partial groups. Then, the following conditions are equivalent for any map $f : S \to T$.*

(i) f *is* $\wp$*-continuous*
(ii) $f^{-1}[U]$ *is a* $\wp$*-open set in* S *for each* $\wp$*-open set* U *in* T.
(iii) $f^{-1}[U]$ *is a* $\wp$*-closed set in* S *for each* $\wp$*-closed set* U *in* T.

Proof (i)→ (ii) Let f be $\wp$-continuous and let $U \subseteq T$ be $\wp$-open. So, $h^{-1}[f^{-1}[U]] = (fh)^{-1}[U]$ is open in C, for each $\wp$-test map $h : C \to T$. Hence, $f^{-1}[U]$ is a $\wp$-open set in S.

(ii)→ (iii) Let U be $\wp$-closed in T. So $T - U$ is $\wp$-open in T. Therefore, $f^{-1}[T - U] = S - f^{-1}[U]$ is $\wp$-open in S. Hence, $f^{-1}[U]$ is $\wp$-closed in S.

(iii)→ (ii) Let U be $\wp$-open in T. So, $T - U$ is $\wp$-closed in T. Therefore, $f^{-1}[T - U] = S - f^{-1}[U]$ is $\wp$-closed in S. Hence, $f^{-1}[U]$ is $\wp$-open in S.

(iii)→ (i) Let $h : C \to S$ be a $\wp$-test map and $U \subseteq T$ be open. So, $f^{-1}[U]$ is $\wp$-open in S. Therefore, $h^{-1}[f^{-1}[U]] = (fh)^{-1}[U]$ is open in C. Hence, f is $\wp$-continuous. □

Definition 21 *Let S and T be topological partial groups. Then, the map $f : S \to T$ is called $\wp$-open if $f(U)$ is $\wp$-open in T for each $\wp$-open set U in S. Also, the map $f : S \to T$ is called $\wp$-closed if $f(U)$ is $\wp$-closed in T for each $\wp$-closed set U in S.*

Theorem 12 *If $f_1 : S_1 \to T_1$ and $f_2 : S_2 \to T_2$ are $\wp$-open maps, then $f_1 \times f_2 : S_1 \times S_2 \to T_1 \times T_2$ is also a $\wp$-open map.*

Proof Let $U \subseteq S_1 \times T_1$ be $\wp$-open and $(x, y) \in U$. Then, there exists a $\wp$-basic neighbourhood $M \times N$ of (x, y) such that $(x, y) \in \wp - (M \times N) \subseteq U$. So, $(f_1 \times f_2)[M \times N] \subseteq (f_1 \times f_2)[U]$. Therefore, $f_1[M] \times f_2[N] \subseteq (f_1 \times f_2)[U]$. Since f_1 and f_2 are $\wp$-open maps, then $f_1[M]$ and $f_2[N]$ are $\wp$-open sets in T_1 and T_2, respectively. Hence, $f_1 \times f_2$ is $\wp$-open. □

Theorem 13 *The maps r_a and l_a are $\wp$-open maps.*

Proof We only prove that r_a is $\wp$-open as follows: Let $U \subseteq S$ be $\wp$-open. Then, $U \bigcap S_{e_x}$ is open in the maximal topological subgroup S_{e_x} and so is open in S. Now, we have two cases:

(i) Let $r_a|_{S_{e_x}} : S_{e_x} \to S_{e_y}$. So, $r_a|_{S_{e_x}} (U \bigcap S_{e_x}) = Ua \bigcap S_{e_y}$. We show that $Ua \bigcap S_{e_y}$ is open in S as follows: Let $h : C \to S$ be a $\wp$-test map. Then, $r_a h : C \to S$ is a $\wp$-test map. Now, $(r_a h)^{-1}(Ua \bigcap S_{e_y}) = h^{-1}((r_a)^{-1}(Ua \bigcap S_{e_y})) = h^{-1}((r_a)^{-1}(Ua) \bigcap (r_a)^{-1}(S_{e_y})) = h^{-1}(U \bigcap S_{e_x})$. Since $U \bigcap S_{e_x}$ is open in S, then $h^{-1}(U \bigcap S_{e_x})$ is open in C. Hence, $Ua \bigcap S_{e_y}$ is $\wp$-open in S.
(ii) Let $r_a|_{S_{e_x}} : S_{e_x} \to S_{e_x}$. Since, the right transformation $r_a|_{S_{e_x}}$ is a homeomorphism of the topological maximal subgroups S_{e_x} , then $r_a|_{S_{e_x}} (U \bigcap S_{e_x})$ is open in S_{e_x}. Since S_{e_x} is open in S, then $r_a|_{S_{e_x}} (U \bigcap S_{e_x}) = Ua \bigcap S_{e_x}$ is open in S. That means $r_a(U) = \bigcup_{e_x \in E(S)} r_a|_{S_{e_x}} (U \bigcap S_{e_x})$ is $\wp$-open in S.

Similarly, we can prove that ℓ_a is $\wp$-open. □

Theorem 14 *Let S be a topological partial group and $A, B \subseteq S$. Then, if A is $\wp$-open in S, then AB and BA are also $\wp$-open in S.*

Proof We only prove that AB is $\wp$-open in S as follows: Since $AB = \bigcup_{b \in B} r_b(A)$, and $r_b(A)$ is $\wp$-open in S, then AB is $\wp$-open in S. Similarly, we can prove that BA is also $\wp$-open in S. □

Theorem 15 *If S is a topological partial group, then every $\wp$-open topological subpartial group of S is $\wp$-closed.*

Proof Let A be a $\wp$-open topological subpartial group of S. Then, xA is $\wp$-open in S, for all $x \in S$. Since $S - A = \bigcup_{x \neq A} xA$, then $S - A$ is $\wp$-open. Therefore, A is $\wp$-closed. □

Theorem 16 *The projection maps $P_1 : S \times T \to S$ and $P_2 : S \times T \to T$ are $\wp$-open maps.*

Proof we only prove that P_1 is $\wp$-open, as follows: let $W \subseteq S \times T$ be $\wp$-open and $x \in P_1[W]$. Then, there exists $y \in T$ such that $(x, y) \in W$. Since W is $\wp$-open, then there exists a $\wp$-basic neighbourhood $M \times N$ of (x, y) such that $(x, y) \in M \times N \subseteq W$. So, $x \in M = P_1^{-1}[M \times N] \subseteq P_1[W]$. Hence, $P_1[W] \in \wp - N_x$. Therefore, P_1 is $\wp$-open. Similarly, we can prove that P_2 is $\wp$-open. □

Let $\{S_i : i = 1, 2, \cdots, n\}$ be a family of topological partial groups and $S = \overset{n}{\underset{i=1}{\otimes}} S_i$ be the cartesian product of topological partial groups. That is, $S = \{x = \langle x_i \rangle : x_i \in S_i, \forall\, i = 1, 2, \ldots, n\}$.

Theorem 17 *The partial group S with the cartesian product topology $S = \overset{n}{\underset{i=1}{\otimes}} S_i$ is a topological partial group.*

Proof The maps μ, γ and e_S are $\wp$-continuous, since $\mu = \langle \mu_i (P_i \times P_i) \rangle$, $\gamma = \langle \gamma_i P_i \rangle$ and $e_S = \langle e_{S_i} P_i \rangle$, respectively, where $P_i : \overset{n}{\underset{i=1}{\otimes}} (S_i) \to S_i$, are the projection maps. □

Definition 22 *Let S and T be topological partial groups. A topology $\wp - \tau^*$ on T is called $\wp-$final with respect to the map $f : S \to T$ if, for any topological partial group F and all maps $g : T \to F$, we have that g is $\wp$-continuous if $gf : S \to F$ is $\wp$-continuous.*

Theorem 18 *The $\wp - \tau^*$ final topology on T with respect to the function $f : S \to T$ exists and is characterized by the following condition: If $U \subseteq T$, then U is $\wp$-open ($\wp$-closed) in T if and only if $f^{-1}[U]$ is $\wp$-open ($\wp$-closed) in S.*

Proof It is clear that ϕ and T are $\wp$-open sets in S. If U and V are $\wp$-open sets in T, then $f^{-1}[U \bigcap V] = f^{-1}[U] \bigcap f^{-1}[V]$ is $\wp$-open in S. So, $U \bigcap V$ is $\wp$-open in T. Similarly, let $(U_\lambda)_{\lambda \in L}$ be a subfamily of $\wp$-open sets in T. Then, $f^{-1}[\bigcup (U_\lambda)]$ are $\wp$-open sets in S. So, $\bigcup U_\lambda$ is a $\wp$-open set in S. A similar proof applies with $\wp$-open replaced by $\wp$-closed. □

Definition 23 *Let S and T be topological partial groups. Then, the map $f : S \to T$ is called $\wp$-identification iff f is surjective and T has the $\wp$-final topology with respect to f.*

Theorem 19 *Let $f : S \to T$ be a $\wp$-continuous surjection. If f is a $\wp$-open (closed) map. Then, f is a $\wp$-identification map.*

Proof Let $U \subseteq T$ be a $\wp$-open set. Then, $f^{-1}[U]$ is $\wp$-open in S. Since f is surjective, then $f[f^{-1}[U]] = U$. Hence, $f^{-1}[U]$ is $\wp$-open in S if and only if U is $\wp$-open. A similar proof applies with open replaced by $\wp$-closed. □

Quotients in topological partial groups

Definition 24 *If S is a topological partial group and $N \leq S$, then S/N with the $\wp$-identification topology, with respect to the quotient map $\rho_N : S \to S/N$, is called the $\wp$-coset space.*

Theorem 20 *Let S be a topological partial group and $N \leq S$. Then, the quotient map $\rho_N : S \to S/N$ is $\wp$-open.*

Proof Let $U \subseteq S$ be open. Then,

$$\begin{aligned}\rho_N^{-1}(\rho_N(U)) &= \{x \in S : \rho_N(x) \in \rho_N(U)\}\\ &= \{x \in S : xN \in U/N\}\\ &= \{x \in S : x \in aN \ \text{ for some } \ a \in U\}\\ &= \textstyle\bigcup_{a \in U} aN\\ &= UN.\end{aligned}$$

Since U is open in S, then UN is open in S. Since ρ_N is an identification map and UN is open in S , then $\rho_N(U)$ is open is S/N. □

Theorem 21 *If S is a topological partial group and $N \trianglelefteq S$, then S/N is a topological partial group.*

Proof Since ρ_N is a $\wp$-open identification map, then $\rho_N \times \rho_N$ is a $\wp$ identification map. So, the product map $\mu : S/N \times S/N \to S/N$ is continuous, since $\mu\,(\rho_N \times_k \rho_N) = \rho_N\,\mu'$, where $\mu' : S \times_k S \to S$ is the product map. The partial inverse map $\gamma : S/N \to S/N$and the partial identity map $e_{S/N} : S/N \to S/N$ are continuous, since $\gamma\,\rho_N = \rho_N\,\gamma'$ and $e_{S/N}\,\rho_N = \rho_N\,e_S$ are $\wp$-continuous and ρ_N is an identification map, where $\gamma' : S \to S$, $x \mapsto x^{-1}$ and $e_S : S \to S$, $x \mapsto e_x$ are $\wp$-continuous. □

Theorem 22 *Let $\varphi : S \to T$ be an idempotent separating surjective $\wp$-morphism and $K = ker\varphi$. Then, there exists a unique bijective $\wp$-morphism $\alpha : S/K \to T$ such that $\varphi = \alpha\rho_K$.*

Proof It is clear that α is bijective and a partial group homomorphism. Also, α is $\wp$-continuous since φ is $\wp$-continuous and ρ_K is a $\wp$-identification map. □

Theorem 23 *Let S be a topological partial group and $M, N \trianglelefteq S$ such that $M \subseteq N$, then*

(i) $N/M \trianglelefteq S/M$
(ii) *There exists a unique bijective $\wp$-morphism $\alpha : (S/M)/(N/M)$ such that* $\rho_N = \alpha\rho_{N/M}\rho_M$

Proof (i) See [4]

(ii) Let $\rho_N : S \to S/N$ and $\rho_N : S \to S/M$ be the quotient maps. Then, ρ_N is an idempotent separating surjective $\wp$-morphism and $ker\rho_N = N$. So, from the last theorem, there exists a unique bijective $\wp$-morphism $\varphi : S/M \to S/N$ such that $\varphi\rho_M = \rho_N$. Since $ker\varphi = N/M$ is a topological partial group, then by the last theorem, there exists a unique bijective $\wp$-morphism $\alpha : (S/M)/(N/M)$, such that $\rho_N = \alpha\rho_{N/M}\rho_M$. □

Separation axioms.

Definition 25 *Let S be a topological partial group. Then, S is called $\wp$-Hausdorff if, for all $x, y \in S$, there exist $\wp$-open sets U and V such that $x \in U$, $y \in V$, and $U \bigcap V \neq \phi$.*

Theorem 24 *Let S be a topological partial group. Then, S is Hausdorff if and only if S is a T_0-space.*

Proof Let S be a Hausdorff partial group. Then, S is a T_0-space. Conversely, let S be a T_0-space. Let $x, y \in S, x \neq y$:

(i) If $x, y \in S_a$, then S_a is a T_2-group and there exist two open sets U, V in S_a and also $\wp$-open in S such that $U \bigcap V \neq \phi$ and $x \in U, y \in V$ and

(ii) If $x \in S_a$ and $y \in S_b$, then, we have that S_a and S_b are $\wp$-open and $S_a \bigcap S_b \neq \phi$. So, S is a Hausdorff partial group.

□

Theorem 25 *Let S be a Hausdorff topological partial group. If $f, g : S \to T$ are $\wp$-morphisms of topological partial group, then the difference kernel $A = \{x \in S : f(x) = g(x)\}$ is a $\wp$-closed subpartial group.*

Proof A is closed (see [3]). Let $x, y \in A$. Now,

$$\begin{aligned} f(xy^{-1}) &= f(x) f(y^{-1}) \\ &= f(x) f(y)^{-1} \\ &= g(x) g(y)^{-1} = g(xy^{-1}). \end{aligned}$$

Therefore, $xy^{-1} \in A$. Hence, A is a $\wp$-closed subpartial group. □

Let $\acute{\mathbf{K}}$ be the category of topological partial groups, as objects and the $\wp$-morphisms, as arrows.

The category $\acute{\mathbf{K}}$ is a convenient category since this category has a product and a quotient.

Abbreviations

$\acute{\mathbf{K}}$: The category of topological partial groups, as objects and the $\wp$-morphisms of topological partial groups, as arrows; $\mathbf{K}$: The category of topological spaces, as objects and k-continuous maps, as arrows; **kpg**: The category of $\underline{k}$-partial groups, as objects, and the morphisms of $\underline{k}$-partial groups, as arrows; **Tpg**: The category of topological partial groups, as objects and the homorphisms of topological partial groups, as arrows; $\boldsymbol{\tau}$: The category of topological spaces, as objects and continuous maps, as arrows; $\boldsymbol{\wp}$: A non-empty full subcategory of $\boldsymbol{\tau}$

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