Zeen El Deen *Journal of the Egyptian Mathematical Society* (2019) 27:20
https://doi.org/10.1186/s42787-019-0025-x

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH Open Access

Edge even graceful labeling of some graphs

Mohamed R. Zeen El Deen

Correspondence:
mohamed.zeeneldeen@suezuniv.edu.eg
Department of Mathematics, Faculty of Science, Suez University, Suez 43527, Egypt

Abstract

Edge even graceful labeling is a new type of labeling since it was introduced in 2017 by Elsonbaty and Daoud (Ars Combinatoria 130:79–96, 2017). In this paper, we obtained an edge even graceful labeling for some path-related graphs like Y- tree, the double star $B_{n,m}$, the graph $\langle K_{1,2n} : K_{1,2m} \rangle$, the graph $P_{2n-1} \odot \overline{K_{2m}}$, and double fan graph $F_{2,n}$. Also, we showed that some cycle-related graphs like the prism graph $\prod_n$, the graph $C_n\left(\frac{n}{2}\right)$, the flag FL_n, the graph $K_2 \odot C_n$, the flower graph $FL(n)$, and the double cycle $\{C_{n,n}\}$ are edge even graphs.

Keywords: Edge-even graceful labeling, Flag graph FL_n, Double fan graph $F_{2,n}$, Prism graph, The flower graph $FL(n)$

2010 Mathematics subject classification: 05 C 78, 05 C 76, 05 C 90, 05 C 99

Introduction

A graph labeling is an assignment of integers to the edges or vertices, or both, subject to certain condition. The idea of graph labelings was introduced by Rosa in [1]. Following this paper, other studies on different types of labelings (Odd graceful, Chordal graceful, Harmonious, edge odd graceful) introduced by many others [2–4]. A new type of labeling of a graph called an edge even graceful labeling has been introduced by Elsonbaty and Daoud [5]. They introduced some path- and cycle-related graphs which are edge even graceful.

Graph labelings give us useful models for a wide range of applications such as coding theory, X-ray, astronomy, radar, and communication network addressing.

Definition 1 *[5] An edge even graceful labeling of a graph $G(V(G), E(G))$ with $p = |V(G)|$ vertices and $q = |E(G)|$ edges is a bijective mapping f of the edge set $E(G)$ into the set $\{2, 4, 6, \cdots, 2q\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 2, 4, \cdots, 2q\}$, given by: $f^*(x) = \left(\sum_{xy \in E(G)} f(xy)\right)$ mod $(2k)$, is an injective function, where $k = max(p, q)$. The graph that admits an edge even graceful labeling is called an edge even graceful graph.*

In Fig. 1, we present an edge even graceful labeling of the Peterson graph and the complete graph K_5.

Edge even graceful for some path related graphs

A Y- tree is a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point, and it is denoted by Y_n where n is the number of vertices in the tree.

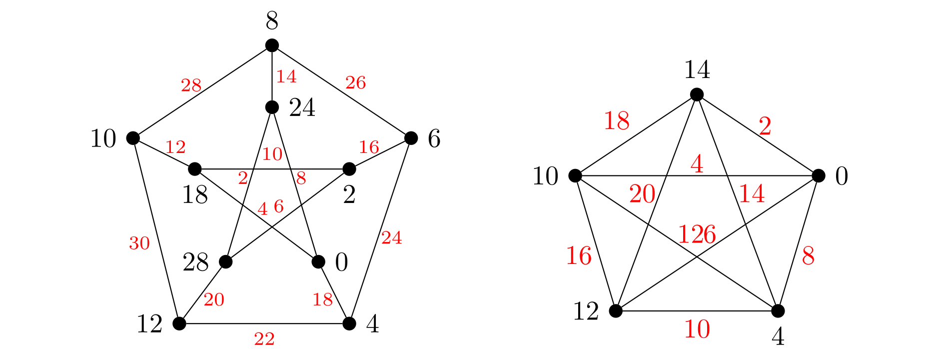

Fig. 1 An edge even graceful labeling of Peterson graph and the complete graph K_5

Lemma 1 *The Y-tree* Y_n *is an edge even graceful graph when* n *is odd.*

Proof The number of vertices of Y-tree Y_n is n and the number of edges is $n-1$. Let the vertices and the edges of Y_n be given as in Fig. 2.

We define the mapping $f : E(Y_n) \rightarrow \{2, 4, \cdots, 2n-2\}$ as follows:

$$
\begin{aligned}
&f(e_i) = 2i && \text{for} \quad 1 \le i \le \tfrac{n+1}{2}\\
&f\left(e_{\frac{n+3}{2}}\right) = n+5, && f\left(e_{\frac{n+5}{2}}\right) = n+3\\
&f(e_i) = 2i && \text{for} \quad \tfrac{n+7}{2} \le i \le n-1
\end{aligned}
$$

Then, the induced vertex labels are

$$f^*(v_1) = 2, \qquad f^*(v_2) = 2, \quad f^*(v_3) = 12$$

$$f^*(v_i) = 4i - 2 \qquad \text{for} \quad 4 \le i \le \frac{n-1}{2},$$

$$f^*\left(v_{\frac{n+1}{2}}\right) = \left[f\left(e_{\frac{n-1}{2}}\right) + f\left(e_{\frac{n+1}{2}}\right)\right] \bmod (2n) = (2n) \bmod (2n) \; = 0$$

Similarly, $f^*\left(v_{\frac{n+3}{2}}\right) = 6$, $\quad f^*\left(v_{\frac{n+5}{2}}\right) = 8$, $\quad f^*\left(v_{\frac{n+7}{2}}\right) = 10$,

$$f^*(v_i) = (4i-2) \bmod (2n) \quad \text{for} \quad \frac{n+9}{2} \le i \le n-1,$$

$$f^*(v_n) = 2n - 2$$

Clearly, all the vertex labels are even and distinct. Thus, Y-tree Y_n is an edge even graceful graph when n is odd. □

Illustration: The edge even labeling of the graph Y_{13} is shown in Fig. 3.

Double star is the graph obtained by joining the center of the two stars $K_{1,n}$ and $K_{1,m}$ with an edge, denoted by $B_{n,m}$. The graph $B_{n,m}$ has $p = n + m + 2$ and $q = n + m + 1$.

Lemma 2 *The double star* $B_{n,m}$ *is edge even graceful graph when one (* m *or* n*) is an odd number and the other is an even number.*

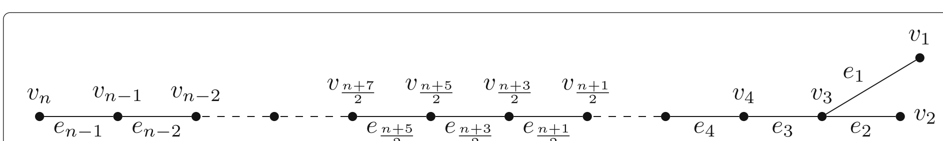

Fig. 2 Y_n with ordinary labeling

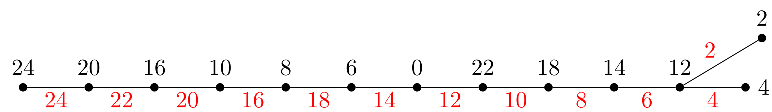

Fig. 3 The edge even labeling of the graph Y_{13}

Proof Without loss of generality, assume that n is odd and m is even. Let the vertex and edge symbols be given as in Fig. 4.

Define the mapping $f : E(B_{m,n}) \to \{2, 4, \cdots, 2q\}$ as follows:

$$f(e_i) = 2i \quad \text{for} \quad 1 \le i \le \frac{n+1}{2}$$
$$f(E_i) = 2p - 2(i+1) \quad \text{for} \quad 1 \le i \le \frac{n-1}{2}$$
$$f(a_i) = m + 2i \quad \text{for } 1 \le i \le \frac{m}{2}$$
$$f(b_i) = 2p - [m + 2i] \quad \text{for } 1 \le i \le \frac{m}{2}$$
$$f(w) = 2p - 2$$

We realize the following:

$$f(E_i) + f(e_{i+1}) \equiv 0 \bmod (2p) \quad \text{for} \quad i = 1, 2, \cdots, \frac{n-1}{2}$$
$$f(a_i) + f(b_i) \equiv 0 \bmod (2p) \quad \text{for} \quad j = 1, 2, \cdots, \frac{m}{2}$$

So, the vertex labels will be

$$f^*(u) = [f(e_1) + f(w)] \bmod (2p) = 0 \quad \text{and}$$
$$f^*(v) = f(w) = (2p-2) \bmod (2p) = 2p-2$$

Also, each pendant vertex takes the labels of its incident edge which are different from the labels of the vertices U and v. □

The graph $\langle K_{1,n} : K_{1,m} \rangle$ is obtained by joining the center v_1 of the star $K_{1,n}$ and the center v_2 of the another star $K_{1,m}$ to a new vertex u, so the number of vertices is $p = n + m + 3$ and the number of edges is $q = n + m + 2$.

Lemma 3 *The graph $\langle K_{1,2n} : K_{1,2m} \rangle$ is an edge even graceful graph.*

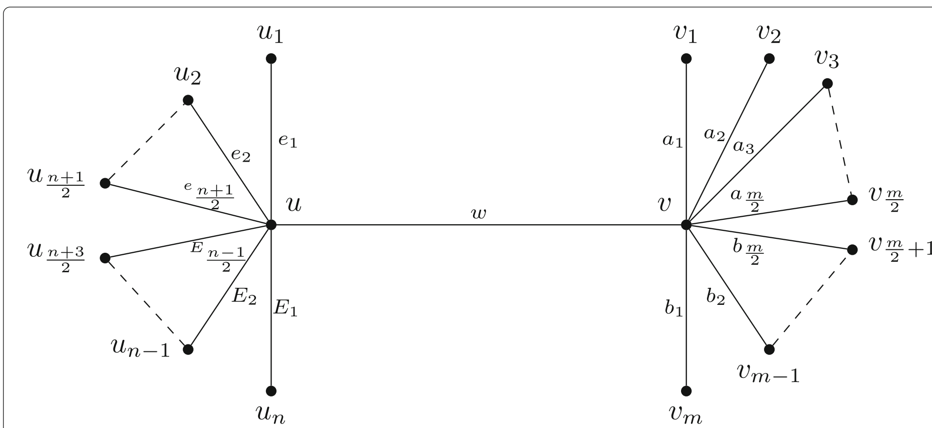

Fig. 4 $B_{m,n}$ with ordinary labeling

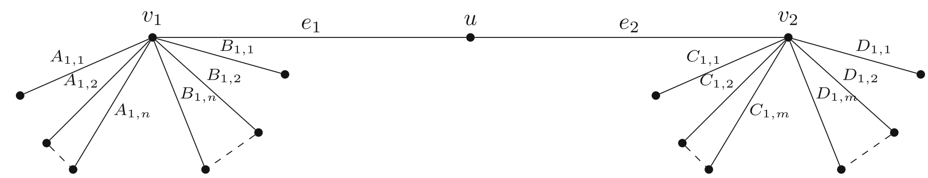

Fig. 5 $\langle K_{1,2n} : K_{1,2m} \rangle$ with ordinary labeling

Proof Let the vertex and edge symbols be given as in Fig. 5.

Define the mapping $f : E(G) \rightarrow \{2, 4, \cdots, 2p\}$ as follows:

$$\begin{aligned}
&f(e_1) = mn + 2 \quad , \quad f(e_2) = 2p - (mn + 2) \\
&f(A_i) = 2i \quad \text{for} \quad i = 1, 2, \cdots, n \\
&f(B_i) = 2p - 2i \quad \text{for} \quad i = 1, 2, \cdots, n \\
&f(C_i) = 2n + 2i \quad \text{for} \quad i = 1, 2, \cdots, m \\
&f(B_i) = 2p - [2n + 2i] \quad \text{for} \quad i = 1, 2, \cdots, m
\end{aligned}$$

We can see the following:

$$\begin{aligned}
&f(A_i) + f(B_i) \equiv 0 \bmod (2p) \quad \text{for} \quad i = 1, 2, \cdots, n \\
&f(C_i) + f(D_i) \equiv 0 \bmod (2p) \quad \text{for} \quad j = 1, 2, \cdots, m
\end{aligned}$$

So, the vertex labels will be

$$\begin{aligned}
&f^*(u) = [f(e_1) + f(e_2)] \bmod (2p) = 0, \\
&f^*(v_1) = f(e_1) = mn + 2 \text{ and} \\
&f^*(v_2) = f(e_2) = 2p - [2n + 2i]
\end{aligned}$$

which are even and distinct from each pendant vertices. Thus, the graph $\langle K_{1,2n} : K_{1,2m} \rangle$ is an edge even graceful graph. □

Lemma 4 *The corona* [6] $P_3 \odot \overline{K_{2m}}$ *is an edge even graceful graph.*

Proof In this graph $p = 6m + 3$ and $q = 6m + 2$. Let the vertex and edge symbols be given as in Fig. 6.

We can define the mapping $f : E(P_3 \odot \overline{K_{2m}}) \rightarrow \{2, 4, \cdots, 2q\}$ as follows:

$$\begin{aligned}
&f(e_i) = 6m + 2i \quad \text{for} \quad i = 1, 2 \\
&f(A_{ij}) = 2(i-1)m + 2j \quad \text{for} \quad i = 1, 2, \cdots, m \\
&f(B_{ij}) = 2p - [2(i-1)m + 2j] \quad \text{for} \quad i = 1, 2, \cdots, m
\end{aligned}$$

It is clear that $f(A_{ij}) + f(B_{ij}) \equiv 0 \bmod (2p)$ for $j = 1, 2, \cdots, m$

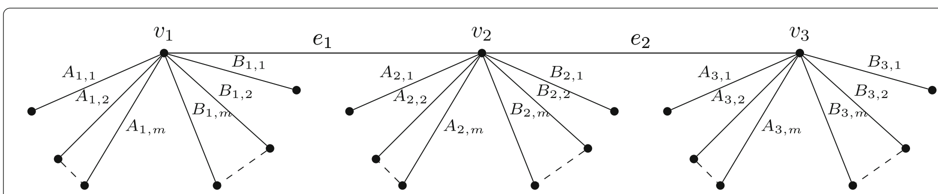

Fig. 6 $P_3 \odot \overline{K_{2m}}$ with ordinary labeling

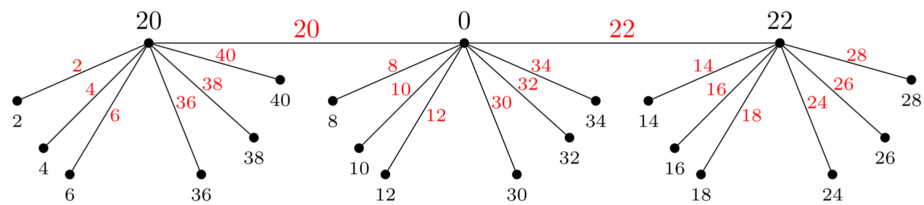

Fig. 7 Edge even graceful labeling of the graph $P_3 \odot \overline{K_6}$

So the vertex labels will be,

$$f^*(v_1) = f(e_1) = 6m + 2 \quad , \qquad f^*(v_3) = f(e_2) = 6m + 4 \quad \text{and}$$

$$f^*(v_2) = [\,f(e_1) + f(e_2)]\ \text{mod}\ (2p) = [\,(6m+2) + (6m+4)]\ \text{mod}\ (2p) = 0$$

Therefore, all the vertices are even and distinct which complete the proof. □

Illustration: The edge even labeling of the graph $P_3 \odot \overline{K_6}$ is shown in Fig. 7.
The generalization of the previous result is presented in the following theory

Theorem 1 *The graph* $P_{2n-1} \odot \overline{K_{2m}}$ *is an edge even graceful graph.*

Proof In this graph, $p = 4nm + 2(n-m) - 1$ and $q = 4nm + 2(n-m) - 2$. The middle vertex in the path P_{2n-1} will be v_n , and we start the labeling from this vertex. Let the vertex and edge symbols be given as in Fig. 8.

Define the mapping $f : E(G) \to \{2, 4, \cdots, 2q\}$ by the following arrangement

$f(e_i) = 2^i$ for $i = 1, 2, \cdots, n-1$

$f(b_i) = 2p - 2^i$ for $i = 1, 2, \cdots, n-1$

$f(A_{ij})$ will take any number from the reminder set of the labeling not contains 2^i nor $2p - 2^i$ for $j = 1, 2, \cdots, m$ and

$f(D_{ij}) = 2p - [f(A_{ij})]$ for $j = 1, 2, \cdots, m.$

It is clear that $[f(A_{ij}) + f(D_{ij})]\ \text{mod}\ (2p) \equiv 0\ \text{mod}\ (2p)$ for $j = 1, 2, \cdots, m.$

So, the vertex labels will be

$f^*(v_n) = [f(e_1) + f(b_1)]\ \text{mod}\ (2p) \equiv 0\ \text{mod}\ (2p)$,

For any vertex v_k when $k < n$, let $k = n - i$ and $i = 1, 2, \cdots, n-2$

$$f^*(v_k) = f^*(v_{n-i}) = \begin{cases} \left[f(e_i) + f(b_{i+1})\right]\ \text{mod}\ (2p) = f(b_i) & \text{if } i \text{ is odd} \\ \left[f(b_i) + f(e_{i+1})\right]\ \text{mod}\ (2p) = f(e_i) & \text{if } i \text{ is even} \end{cases}$$

When $k > n$, let $k = n + i$ and $i = 1, 2, \cdots, n-2$

$$f^*(v_k) = f^*(v_{n+i}) = \begin{cases} \left[f(b_i) + f(e_{i+1})\right]\ \text{mod}\ (2p) = f(e_i) & \text{if } i \text{ is odd} \\ \left[f(e_i) + f(b_{i+1})\right]\ \text{mod}\ (2p) = f(b_i) & \text{if } i \text{ is even} \end{cases}$$

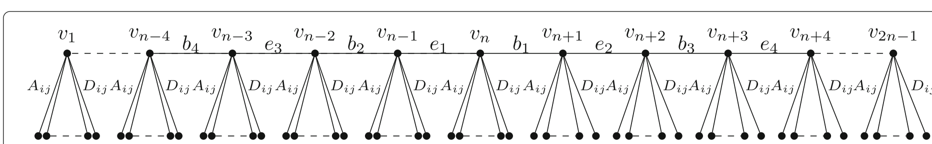

Fig. 8 $P_{2n-1} \odot \overline{K_{2m}}$ with ordinary labeling

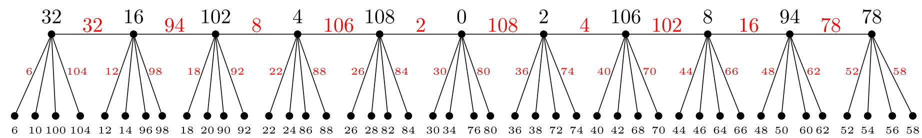

Fig. 9 Edge even graceful labeling of the graph $P_{11} \odot \overline{K_4}$

The pendant vertices v_1 *and* v_{2n-1} of the path P_{2n-1} will take the labels of its pendant edges of P_{2n-1},i.e.,

If n is even, then $f^*(v_1) = f(e_{n-1})$, and $f^*(v_{2n-1}) = f(b_{n-1})$

If n is odd, then $f^*(v_1) = f(b_{n-1})$, and $f^*(v_{2n-1}) = f(e_{n-1})$

Then, the labels of the vertices of the path P_{2n-1} takes the labels of the edges of the path P_{2n-1} , and each pendant vertex takes the labels of its incident edge. Then, there are no repeated vertex labels, which complete the proof. □

Illustration: The graph $P_{11} \odot \overline{K_4}$ labeled according to Theorem 1 is presented in Fig. 9.

A double fan graph $F_{2,n}$ is defined as the graph join $\overline{K_2} + P_n$ where $\overline{K_2}$ is the empty graph on two vertices and P_n be a path of length n.

Theorem 2 *The double fan graph $F_{2,n}$ is an edge even graceful labeling when n is even.*

Proof In the graph $F_{2,n}$ we have $p = n + 2$ *and* $q = 3n - 1$. Let the graph $F_{2,n}$ be given as indicated in Fig. 10.

Define the edge labeling function $f : E(F_{2,n}) \longrightarrow \{2, 4, \cdots, 6n - 2\}$ as follows:

$f(a_i) = 2i; \qquad i = 1, 2, \cdots n$

$f(b_i) = 2q - 2i = 6n - 2(i + 1); \qquad i = 1, 2, \cdots n$

$$f(e_i) = \begin{cases} 2n + 2i & \text{if } 1 \le i < \frac{n}{2} \\ 6n - 2 & \text{if } i = \frac{n}{2} \\ 2n + 2(i - 1) & \text{if } \frac{n}{2} < i \le n - 1 \end{cases}$$

Hence, the induced vertex labels are

$f^*(u) = \left(\sum_{i=1}^{n}(f(a_i))\right) \bmod(6n-2) = \left(\sum_{i=1}^{n}(2i)\right) \bmod(6n-2) = (n^2+n) \bmod(6n-2)$

$f^*(v) = \left(\sum_{i=1}^{n}(f(b_i))\right) \bmod(6n - 2) = \left(\sum_{i=1}^{n}(2q - 2i)\right) \bmod(6n - 2) = \left(-n^2 - n\right) \bmod(6n - 2)$

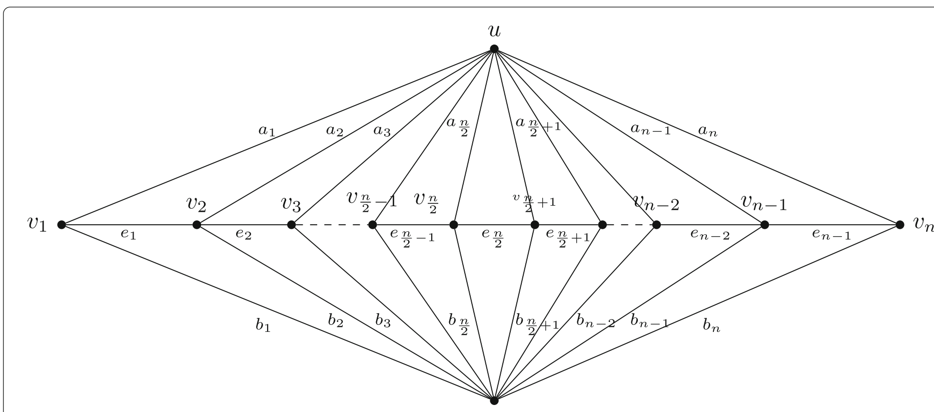

Fig. 10 $F_{2,n}$ with ordinary labeling

$f^*(v_i) = \left[\sum_{i=1}^{n}(f(a_i)+f(b_i)+f(e_i)+f(e_{i-1}))\right] \bmod(6n-2) = (4n+4i-2) \bmod(6n-2)$, $2 \le i \le \frac{n}{2}-1$

$f^*(v_i) = \left[\sum_{i=1}^{n}(f(a_i)+f(b_i)+f(e_i)+f(e_{i-1}))\right] \bmod(6n-2) = (4n+4i-6) \bmod(6n-2)$, $\frac{n}{2}+2 \le i \le n-1$

Since $[f(a_i)+f(b_i)] \bmod(6n-2) = 0$, we see that

$f^*(v_1) = f(e_1) = 2n+2,$

$f^*(v_n) = f(e_{n-1}) = 4n-4,$

$f^*(v_{\frac{n}{2}}) = f(e_{\frac{n}{2}-1}) = 3n-2$ and

$f^*(v_{\frac{n}{2}+1}) = f(e_{\frac{n}{2}}) = 3n$

Thus, the set of vertices $v_1, v_2, v_3, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, v_{\frac{n}{2}+3}, \cdots, v_{n-2}, v_{n-1}, v_n$ are labeled by $2n+2,\ 4n+6,\ 4n+10, \cdots, 6n-6,\ 3n-2,\ 3n,\ 4, 8,\ \cdots, 2n-12,\ 2n-8,\ 4n-4$ respectively.

Clearly, $f^*(u)$ and $f^*(v)$ are different from all the labels of the vertices. Hence $F_{2,n}$ is an edge even graceful when n is even. □

Illustration: The double fan $F_{2,10}$ labeled according to Theorem 2 is presented in Fig. 11.

Edge even graceful for some cycle related graphs

Definition 2 *For $n \ge 4$, a cycle (of order n) with one chord is a simple graph obtained from an n-cycle by adding a chord. Let the n-cycle be $v_1v_2\cdots v_nv_1$. Without loss of generality, we assume that the chord joins v_1 with any one v_i, where $3 \le i \le n-1$. This graph is denoted by $C_n(i)$.*

Lemma 5 *The graph $C_n\left(\frac{n}{2}\right)$ is an edge even graceful graph if n is even.*

Proof Let $\{v_1, v_2, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, \cdots, v_n\}$ be the vertices of the graph $C_n\left(\frac{n}{2}\right)$, and the edges are $e_i = (v_iv_{i+1})$ for $i \le i \le n-1$ and the chord $e_0 = (v_1v_{\frac{n}{2}})$ connecting the vertex v_1 with $v_{\frac{n}{2}}$ as in Fig. 12.

Here, $p = n$ and $q = n+1$, so $2k = 2q = 2n+2$; first, we label the edges as follows:

$$f(e_i) = 2i+2; \quad i = 0, 1, 2, \cdots n$$

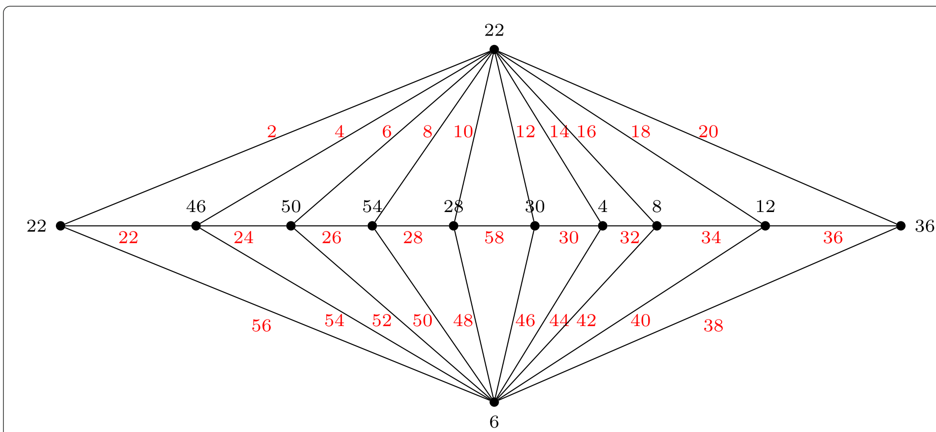

Fig. 11 Edge even graceful labeling of the double fan $F_{2,10}$

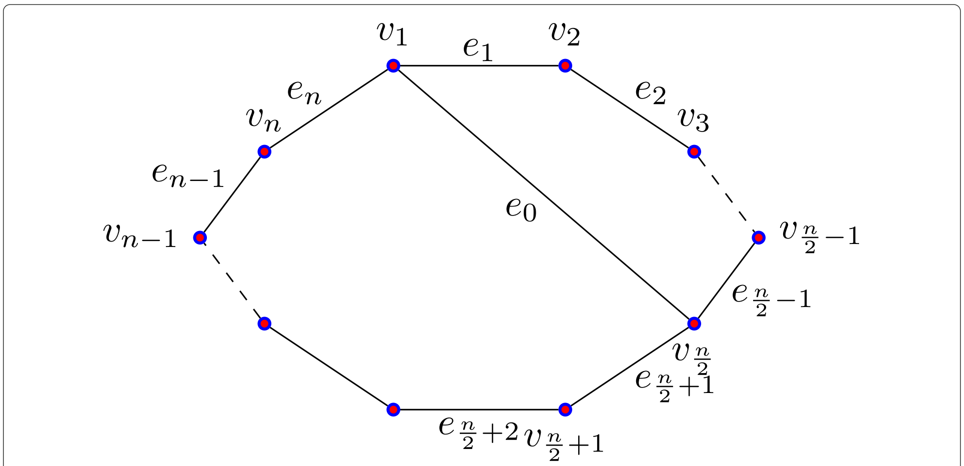

Fig. 12 $C_n\left(\frac{n}{2}\right)$ with ordinary labeling

Then, the induced vertex labels are as follows:

$$f^*(v_1) = \left[f(e_0) + f(e_1) + f(e_n)\right] \bmod(2n+2) = (2+4+2n+2) \bmod(2n+2) = 6,$$
$$f^*(v_{\frac{n}{2}}) = \left[f(e_0 + f(e_{\frac{n}{2}}) + f(e_{\frac{n}{2}-1})\right] \bmod(2n+2) = (2n+4) \bmod(2n+2) = 2$$

for any other vertex v_i, $i \neq 1, \frac{n}{2}$

$$f^*(v_i) = f(e_i) + f(e_{i-1}) = (4n+4) \bmod(2n+2) = (4i+2) \bmod(2n+2)$$

Hence, the labels of the vertices $v_0, v_1, v_2, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, \cdots v_n$ are $6, 10, 14, \cdots, 2n-2, 2, 4, \cdots, 2n$ respectively, which are even and distinct. So, the graph $C_n\left(\frac{n}{2}\right)$ is an edge even graceful graph if n is even. □

Definition 3 *Let C_n denote the cycle of length n. The flag FL_n is obtained by joining one vertex of C_n to an extra vertex called the root, in this graph $p = q = n+1$.*

Lemma 6 *The flag graph FL_n is edge even graceful graph when n is even.*

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the cycle C_n and the edges are $e_i = (v_i v_{i+1})$ *for* $1 \leq i \leq n$ and the edge $e = (v_1 v_0)$ connecting the vertex v_1 with v_0 as in Fig. 13.

First, we label the edges as follows:

$$f(e_i) = 2i + 2; \quad i = 0, 1, 2, \cdots n$$

Then, the induced vertex labels are as follows $f^*(v_0) = f(e_0) = 2$,

$$f^*(v_1) = \left[f(e_0) + f(e_1) + f(e_n)\right] \bmod(2n+2) = (2+4+2n+2) \bmod(2n+2) = 6$$
$$f^*(v_i) = \left[f(e_i) + f(e_{i-1})\right] \bmod(2n+2) = (4i+2) \bmod(2n+2) \quad i = 2, 3, \cdots, n$$

Hence, the labels of the vertices $v_0, v_1, v_2, \cdots, v_{\frac{n}{2}-1}, v_{\frac{n}{2}}, v_{\frac{n}{2}+1}, \cdots v_n$ will be $2, 6, 10, 14, \cdots, 2n-2, 0, 4, \cdots, 2n$ respectively. □

Lemma 7 *The graph $K_2 \odot C_n$ is edge even graceful graph when n is odd.*

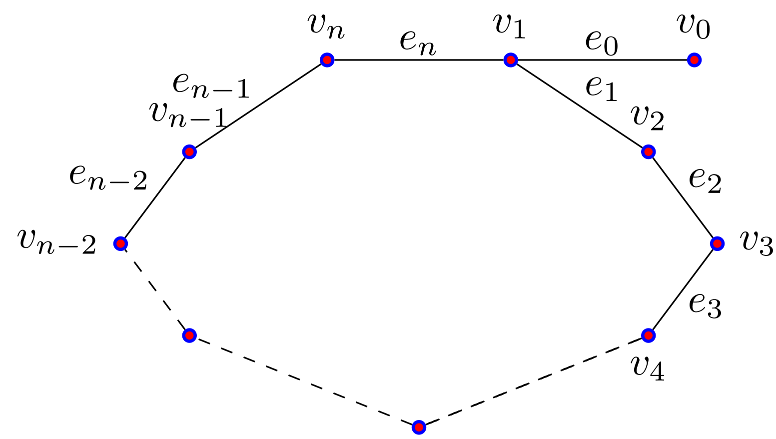

Fig. 13 Fl_n with ordinary labeling

Proof Let $\left\{v_1, v_2, \cdots, v_n, v_1', v_2', \cdots, v_n'\right\}$ be the vertices of the graph $K_2 \odot C_n$ and the edges are $\left\{e_1, e_2, \cdots, e_n, e_1', e_2', \cdots, e_n'\right\}$ as shown in Fig. 14. Here, $p = 2n$ and $q = 2n+1$, so $2k = 4n + 2$.

First, we label the edges as follows:

$$f(e_i) = 2i; \qquad 1 \leq i \leq n+1$$
$$f(e_i') = 2n + 2(i+1); \qquad 1 \leq i \leq n$$

We can see that $[f(e_n) + f(e_{n+1})] \bmod (2q) = (4n+2) \bmod (4n+2) \equiv 0$

Then, the induced vertex labels are as follows

$$f^*(v_i) = [f(e_i) + f(e_{i+1})] \bmod (4n+2) \qquad 1 \leq i \leq n-1$$
$$= [6 + 4(i-1)] \bmod (4n+2), \qquad 1 \leq i \leq n-1$$

$$f^*(v_i') = \left[f(e_i') + f(e_{i+1}')\right] \bmod (4n+2) = [8 + 4(i-1)] \bmod (4n+2), \quad 1 \leq i < n$$
$$f^*(v_n) = [f(e_1) + f(e_n) + f(e_{n+1})] \bmod (4n+2) = f(e_1) \bmod (4n+2) \equiv 2$$
$$f^*(v_n') = \left[f(e_1') + f(e_{n+1}) + f\left(e_n'\right)\right] \bmod (4n+2) \equiv 4$$

Clearly, the vertex labels are all even and distinct. Hence, the graph $K_2 \odot C_n$ is edge even graceful for odd n. □

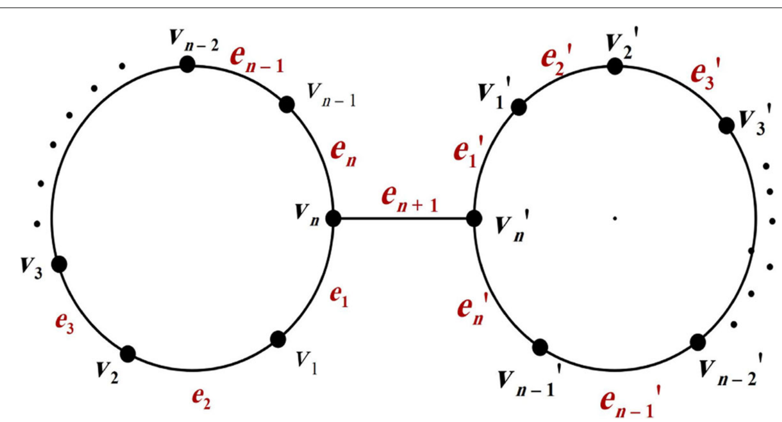

Fig. 14 The graphs $K_2 \odot C_n$ with ordinary labeling

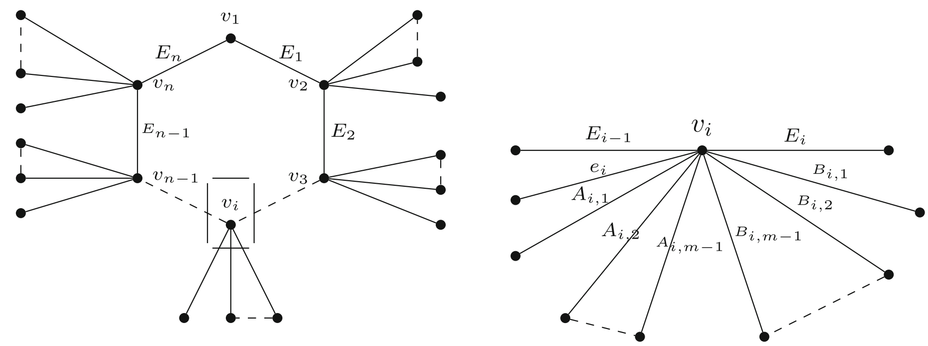

Fig. 15 $\{C_n - \{v_1\}\} \odot \overline{K_{2m-1}}$ with ordinary labeling

Let C_n denote the cycle of length n. Then, the corona of all vertices of C_n except one vertex $\{v_1\}$ with the complement graph $\overline{K_{2m-1}}$ is denoted by $\{C_n - \{v_1\}\} \odot \overline{K_{2m-1}}$, in this graph $p = q = 2m(n-1) + 1$.

Lemma 8 *The graph* $\{C_n - \{v_1\}\} \odot \overline{K_{2m-1}}$ *is an edge even graceful graph.*

Proof Let the vertex and edge symbols be given as in Fig. 15.
Define the mapping $f : E(G) \rightarrow \{2, 4, \cdots, 2q\}$ as follows:

$$f(E_i) = 2(i-1)m + 2 \qquad \text{for} \quad i = 1, 2, \cdots, n-1$$
$$f(E_n) = 2q$$
$$f(e_{i+1}) = 2q - [\,2(i-1)m + 2] \qquad \text{for} \quad i = 1, 2, \cdots, n-1$$
$$f(A_{ij}) = 2(i-1)m + 2j + 2 \qquad \text{for} \quad j = 1, 2, \cdots, m-1$$
$$f(B_{ij}) = 2q - [\,2(i-1)m + 2j + 2] \qquad \text{for} \quad j = 1, 2, \cdots, m-1$$

We realize the following:
$\left[f(A_{ij}) + f(B_{ij})\right] \bmod (2q) \equiv 0 \bmod (2q) \qquad \text{for} \quad j = 1, 2, \cdots, m-1$
Also, $\left[f(E_{i-1}) + f(e_i)\right] \bmod (2q) \equiv 0 \bmod (2q)$ *for* $i = 2, 3, \cdots, n$
So, verifying the vertex labels, we get that,
$f^*(v_1) = \left[f(E_1) + f(E_n)\right] \bmod (2q) = (\,2 + 2q) \bmod (2q) = 2,$

$$f^*(v_i) = \left[\sum_{j=1}^{m-1} f(A_{ij}) + \sum_{j=1}^{m-1} f(B_{ij}) + f(E_i) + f(E_{i-1}) \; + f(e_i)\right] \bmod (2q) \quad i = 2, 3, \cdots, n$$
$$= f(E_i) \bmod (2q) \; = 2(i-1)m + 2 \bmod (2q), \quad i = 2, 3, \cdots, n$$

Hence, the labels of the vertices $v_1, v_2, \cdots, v_n$ takes the label of the edges of the cycles and each of the pendant vertices takes the label of its edge, so they are all even and different numbers. □

Illustration: In Fig. 16, we present an edge even graceful labeling of the graph $\{C_6 - \{v_1\}\} \odot \overline{K_3}$.

Lemma 9 *The double cycle graph* $\{C_{n,n}\}$ *is an edge even graceful graph when n is odd.*

Proof Here, $p = n$ and $q = 2n$. Let the vertex and edge symbols be given as in Fig. 17.

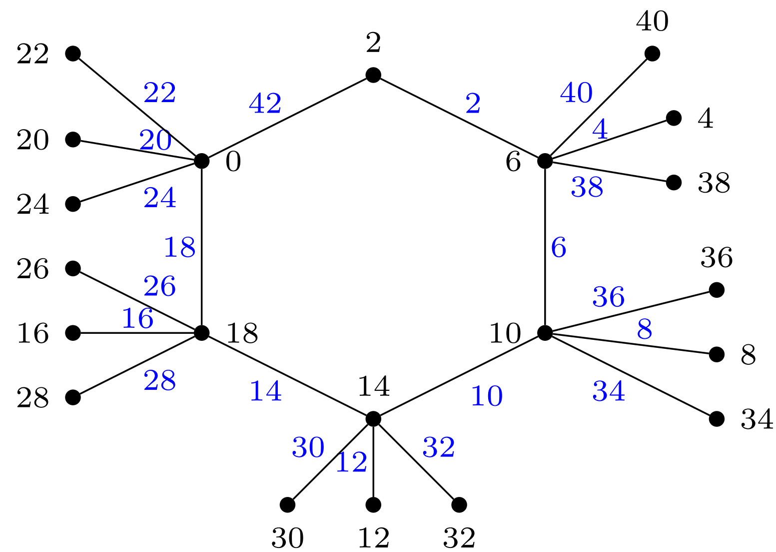

Fig. 16 $\{C_6 - \{v_1\}\} \odot \overline{K_3}$

Define the mapping $f : E(G) \rightarrow \{2, 4, \cdots, 4n\}$ by
$f(e_i) = 2i$ for $i = 1, 2, \cdots, n$. So, the vertex labels will be
$f^*(v_1) = \left[f(e_1) + f(e_n) + f(e_{n+1}) + f(e_{2n})\right] \bmod (4n) = 4$

$$f^*(v_i) = \left[f(e_i) + f(e_{i-1}) + f(e_{i+n}) + f(e_{i+n-1})\right] \bmod (4n) \quad i = 2, , \cdots, n$$
$$= (8i - 4) \bmod (4n) \quad i = 2, 3, \cdots, n$$

Hence, the labels of the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}}, v_{\frac{n+3}{2}}, \cdots, v_n$ will be $4, 12, 20, \cdots, 0, 8, \cdots, 4n - 4$ □

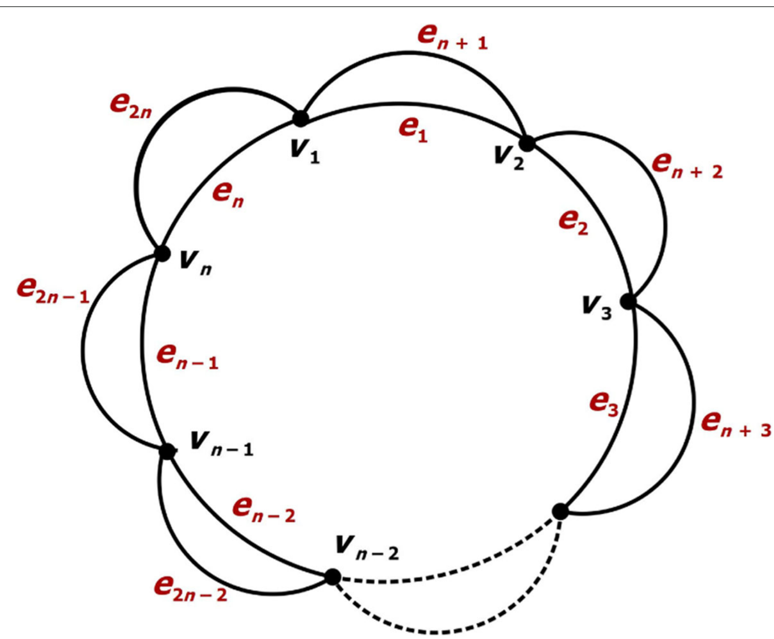

Fig. 17 $\{C_{n,n}\}$ with ordinary labeling

The prism graph $\prod_n$ is the cartesian product $C_n \Box K_2$ of a cycle C_n by an edge K_2, and an n-prism graph has $p = 2n$ vertices and $q = 3n$ edges.

Theorem 3 *The prism graph* $\prod_n$ *is edge even graceful graph.*

Proof In the prism graph $\prod_n$ we have two copies of the cycle C_n, let the vertices in one copy be $v_1, v_2, \cdots, v_n$ and the vertices on the other copy be $v'_1, v'_2, \cdots, v'_n$. In $\prod_n$, the edges will be
$v_i v_{i+1}$, $v'_i v'_{i+1}$, and $v_i v'_i$. Let the vertex and edge symbols be given as in Fig. 18.
Define the mapping $f : E\left(\prod_n\right) \rightarrow \{2, 4, \cdots, 6n\}$ by

$$f(e_i) = 2i \quad \text{for} \quad i = 1, 2, \cdots, n$$
$$f(e'_i) = 4n + 2i \quad \text{for} \quad i = 1, 2, \cdots, n$$
$$f(E_i) = 2n + 2i \quad \text{for} \quad i = 1, 2, \cdots, n$$

So, the vertex labels will be

$$f^*(v_1) = \left[f(e_1) + f(e_n) + f(E_1)\right] \bmod (6n) = 4n + 4$$
$$f^*(v_i) = \left[f(e_i) + f(e_{i-1}) + f(E_{n-i+2})\right] \bmod (6n) = (2i + 4n + 2) \bmod (6n) \quad i = 2, 3, \cdots, n$$

Hence, the labels of the vertices $v_1, v_2, \cdots, v_n$ will be $4n+4,\ 4n+6, \cdots, 0,\ 2$ respectively.
Also, $f^*(v'_1) = \left[f(e'_1) + f(e'_n) + f(E_1)\right] \bmod (6n) = 12n + 4 \bmod (6n) = 4$

$$f^*(v'_i) = \left[f(e'_i) + f(e'_{i-1}) + f\right] \bmod (6n) \quad i = 2,, \cdots, n$$
$$= (2i + 12n + 2) \bmod (6n) = 2i + 2 \quad i = 2, 3, \cdots, n$$

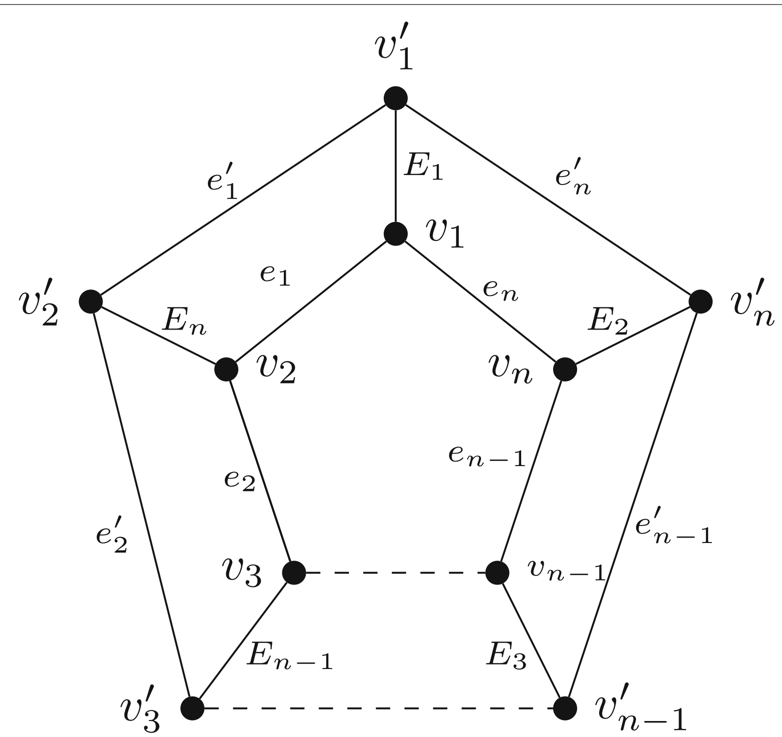

Fig. 18 $\prod_n$ with ordinary labeling

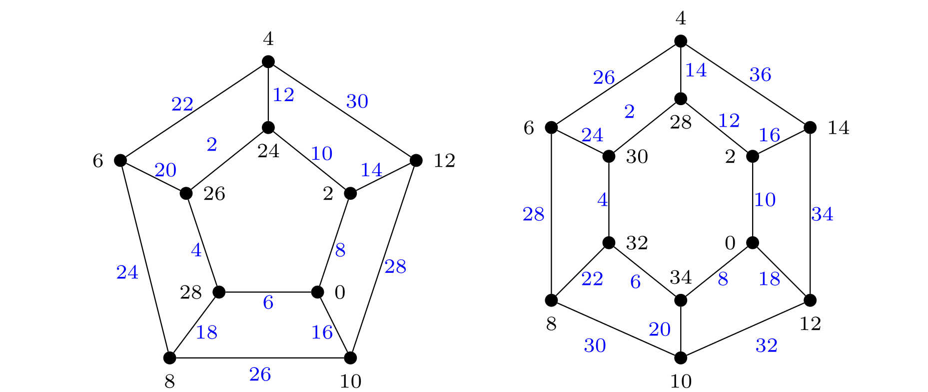

Fig. 19 An edge even graceful labeling of prism graphs $\prod_5$ and $\prod_6$

Hence, the labels of the vertices $v'_1, v'_2, \cdots, v'_n$ are $4,\ 6, 8, \cdots, 2n,\ 2n+2$ respectively. Overall, the vertices are even and different. Thus, the prism graph $\prod_n$ is an edge even graceful graph. □

Illustration: In Fig. 19, we present an edge even graceful labeling of of prism graphs $\prod_5$ and $\prod_6$.

The flower graph FL(n) ($n \geq 3$) is the graph obtained from a helm H_n by joining each pendant vertex to the center of the helm.

Theorem 4 *The flower graph FL(n) ($n \geq 4$) is an edge even graceful graph.*

Proof In the flower graph FL(n) ($n \geq 4$), we have $p = 2n + 1$ and $q = 4n$. Let $\{v_0, v_1, v_2, \cdots, v_n, v'_1, v'_2, \cdots, v'_n\}$ be the vertices of FL(n) and $\{e_1, e_2, e_3, \cdots, e_{3n}, E_1, E_2, E_3, \cdots, E_n\}$ be the edges of FL(n) as in Fig. 20.

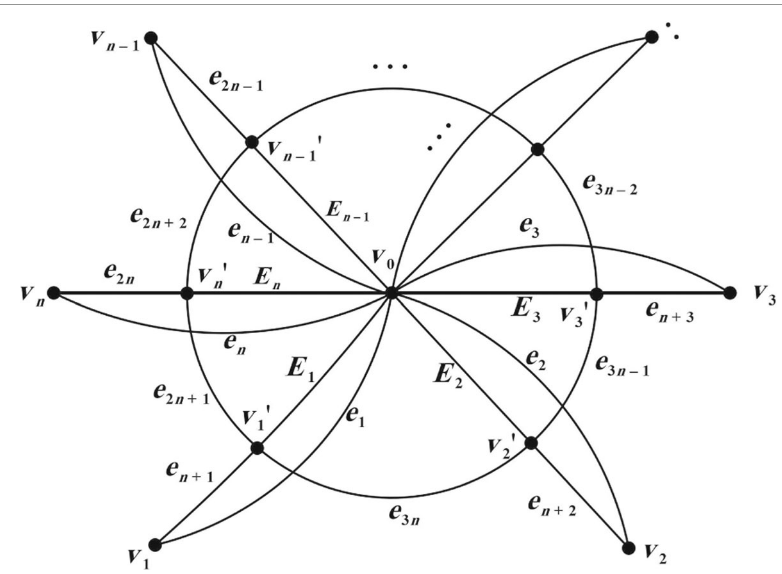

Fig. 20 The flower graph FL(n) with ordinary labeling

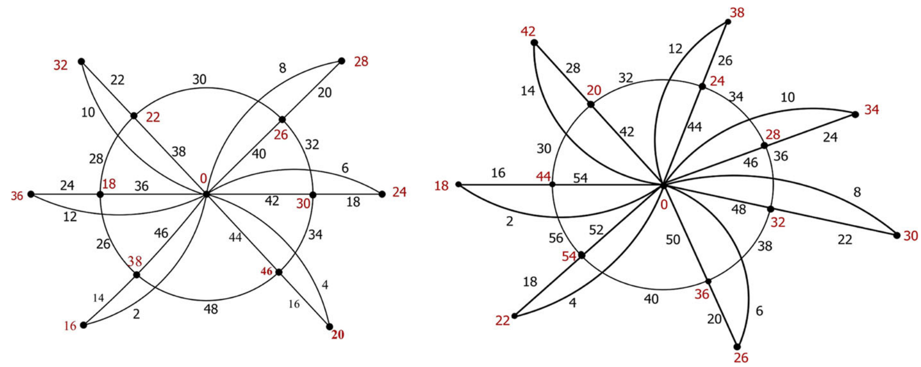

Fig. 21 An edge even graceful labeling of The flower graphs FL(6) and FL(7)

First, define the mapping $f : E(Fl(n)\ ) \rightarrow \{2, 4, \cdots, 8n\ \}$ as the following:

$$f(E_i) = 8n - 2i \qquad \text{for} \quad i = 1, 2, \cdots, n \quad \text{and}$$

$$f\ (e_i) = \begin{cases} 2i & \text{if} \quad 1 \leq i \leq 3n - 1 \\ 8n & \text{if} \quad i = 3n \end{cases}$$

Then, the induced vertex labels are

$$f^*(v_0) = \left[\sum_{i=1}^{n} (f(e_i) + f(E_i) \right] \bmod(8n) = \ 0$$

$$f^*(v_i) = \left[f(e_i) + f(e_{n+i}) \right] \bmod(8n) = \ 2n + 4i\ , \quad i = 1, 2, \cdots, n$$

$$f^*(v'_1) = \left[f(e_{3n}) + f(e_{2n+1}) + f(e_{n+1}) + f(E_1) \right] \bmod(8n) = \ 6n + 2$$

$$f^*(v'_2) = \left[f(e_{3n}) + f(e_{3n-1}) + f(e_{n+2}) + f(E_2) \right] \bmod(8n) = \ 8n - 2$$

$$\begin{aligned} f^*(v'_i) &= \left[f(e_{3n-i+1}) + f(e_{3n-i+2}) + f(e_{n+i}) + f(E_i) \right] \bmod(8n), \qquad 3 \leq i \leq n \\ &= [6(n+1) - 4i] \ \bmod(8n) \qquad 3 \leq i \leq n \end{aligned}$$

Overall, all the vertex labels are even and distinct which complete the proof. □

Illustration: In Fig. 21, we present an edge even graceful labeling of of the flower graphs FL(6)and FL(7).

Acknowledgments
I am so grateful to the reviewers for their valuable suggestions and comments that significantly improved the paper.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Authors' contributions
The author read and approved the final manuscript.

Competing interests
The author declares that he has no competing interests.

Publisher's Note

Received: 10 January 2019 Accepted: 22 February 2019
Published online: 09 July 2019

References

1. Rosa, A.: Certain valuations of the vertices of a graph; Theory of Graphs (Internat. Symp, Rome, July 1966). Gordan and Breach, New York (1967). Paris, France
2. Gallian, J. A.: A Dynamic survey of graph labeling. The Electronic Journal of Combinatorics (2015). http://link.springer.com/10.1007/978-1-84628-970-5
3. Daoud, S. N.: Edge odd graceful labeling of some path and cycle related graphs. AKCE International Journal of Graphs and Combinatorics. **14**, 178–203 (2017). http://dx.doi.org/10.1016/j.akcej.2017.03.001
4. Seoud, M. A., Salim, M. A.: Further results on edge-odd graceful graphs. Turkish J. Math. **40**(3), 647-656 (2016). https://journals.tubitak.gov.tr/math/
5. Elsonbaty, A., Daoud, S. N.: Edge even graceful labeling of some path and cycle related graphs. Ars Combinatoria. **130**(2), 79-96 (2017)
6. Bondy, J. A., Murty, U. S.: Graph Theory. Springer (2008). http://link.springer.com/10.1007/978-1-84628-970-5