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On the joint distribution of order statistics from independent non-identical bivariate distributions

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Abstract

In this note, the exact joint probability density function (*jpdf*) of bivariate order statistics from independent non-identical bivariate distributions is obtained. Furthermore, this result is applied to derive the joint distribution of a new sample rank obtained from the *r*th order statistics of the first component and the *s*th order statistics of the second component.

Keywords: Bivariate order statistics, Joint distribution, Rank, Random vector

Subject classifications: 62G32, 62G30

Introduction

Multivariate order statistics especially Bivariate order statistics have attracted the interest of several researchers, for example, see [1]. The distribution of bivariate order statistics can be easily obtained from the bivariate binomial distribution, which was first introduced by [2]. Considering a bivariate sample, David et al. [3] studied the distribution of the sample rank for a concomitant of an order statistic. Bairamove and Kemalbay [4] introduced new modifications of bivariate binomial distribution, which can be applied to derive the distribution of bivariate order statistics if a certain number of observations are within the given threshold set. Barakat [5] derived the exact explicit expression for the product moments (of any order) of bivariate order statistics from any arbitrary continuous bivariate distribution function (df). Bairamove and Kemalbay [6] used the derived *jpdf* by [5] to derive the joint distribution on new sample rank of bivariate order statistics. Moreover, Barakat [7] studied the limit behavior of the extreme order statistics arising from n two-dimensional independent and non-identically distributed random vectors. The class of limit dfs of multivariate order statistics from independent and identical random vectors with random sample size was fully characterized by [8].

Consider n two-dimensional independent random vectors $\underline{W}_j = (X_j, Y_j), j = 1, 2, ..., n$, with the respective distribution function (df) $F_j(\underline{w}) = F_j(x, y) = P(X_j \leq x, Y_j \leq y), j = 1, 2, ..., n$. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ and $Y_{1:n} \leq Y_{2:n} \leq ... \leq Y_{n:n}$ be the order statistics of the X and Y samples, respectively. The main object of this work is to derive the *jpdf* of the random vector $Z_{k,k':n} = (X_{n-k+1:n}, Y_{n-k'+1:n})$, where $1 \leq k, k' \leq n$. Let $G_j(\underline{w}) = P(\underline{W}_j > \underline{w})$ be the survival function of $F_j(\underline{w}), j = 1, 2, ..., n$ and let $F_{1,j}(.)$, $F_{2,j}(.)$, $G_{1,j}(.) =$

$1-F_{1,j}(.)$ and $G_{2,j}(.) = 1-F_{2,j}(.)$ the marginal dfs and the marginal survival functions of $\Phi_{k,k':n} = P(Z_{k,k':n} \le \underline{w})$, $F_j(\underline{w})$ and $G_j(\underline{w}), j = 1,2,...,n$, respectively. Furthermore, let $F_j^{1,.} = \frac{\partial F_j(\underline{w})}{\partial x}$ and $F_j^{.,1} = \frac{\partial F_j(\underline{w})}{\partial y}$. Also, the *jpdf* of $(X_{n-k+1:n}, Y_{n-k'+1:n})$ is conveniently denoted by $f_{k,k':n}(\underline{w})$. Finally, the abbreviations $\min(a,b) = a \wedge b$, and $\max(a,b) = a \vee b$ will be adopted.

The *jpdf* of non-identical bivariate order statistics

The following theorem gives the exact formula of the jpdf of non-identical bivariate order statistics.

Theorem 1 *The jpdf of non-identical bivariate order statistics is given by*

$$f_{k,k':n}(\underline{w}) = \sum_{\theta,\varphi=0}^{1} \sum_{r=r_{**}}^{r^{**}} \sum_{\rho_{\theta,\varphi,r}} \Pi_{j=1}^{\theta} F_{i_j}^{.,1}(\underline{w}) \Pi_{j=\theta+1}^{1} (f_{2,i_j}(y) - F_{i_j}^{.,1}(\underline{w})) \Pi_{j=2}^{\varphi+1} F_{i_j}^{1,.}(\underline{w})$$
$$\times \Pi_{j=\varphi+2}^{2} (f_{1,i_j}(x) - F_{i_j}^{1,.}(\underline{w})) \Pi_{j=3}^{k-\theta-r+1} (F_{1,i_j}(x) - F_{i_j}(\underline{w})) \Pi_{j=k-\theta-r+2}^{k-\theta+1} F_{i_j}(\underline{w})$$
$$\times \Pi_{j=k-\theta+2}^{k+k'-\theta-\varphi-r} (F_{2,i_j}(y) - F_{i_j}(\underline{w})) \Pi_{j=k+k'-\theta-\varphi-r+1}^{n} G_{i_j}(\underline{w}) + \sum_{r=0\vee(k+k'-n-1)}^{(k-1)\wedge(k'-1)} \sum_{\rho_r} f_j(\underline{w})$$
$$\Pi_{j=2}^{k-r} (F_{1,i_j}(x) - F_{i_j}(\underline{w})) \times \Pi_{j=k-r+1}^{k} F_{i_j}(\underline{w}) \Pi_{j=k+1}^{k+k'-r} (F_{2,i_j}(y) - F_{i_j}(\underline{w})) \Pi_{j=k+k'-r+1}^{n} G_{i_j}(\underline{w}),$$

where $r_{**} = 0 \vee (k+k'-\theta-\varphi-n), r^{**} = (k-\theta-1) \wedge (k'-\varphi-1)$, $\sum_\rho$ *denotes summation subject to the condition* ρ, *and* $\sum_{\rho_{\theta_1,\theta_2,\varphi_1,\varphi_2,\omega,r}}$ *denotes the set of permutations of* $i_1,...,i_n$ *such that* $i_{j_1} < ... < i_{j_n}$.

Proof A convenient expression of $f_{k,k':n}(\underline{w})$ may derived by noting that the compound event $E = \{x < X_{k:n} < x+\delta x, y < Y_{k:n} < y+\delta y\}$ may be realized as follows: $r;\varphi_1;s_1;\theta_1;\omega;\theta_2;s_2;\varphi_2$ and t observations must fall respectively in the regions $I_1 = (-\infty,x] \cap (-\infty,y]\,; I_2 = (x,x+\delta x] \cap (-\infty,y]\,; I_3 = (x+\delta x,\infty] \cap (-\infty,y]\,; I_4 = (-\infty,x] \cap (y,y+\delta y]\,; I_5 = (x,x+\delta x] \cap (y,y+\delta y]\,; I_6 = (x+\delta x,\infty] \cap (y,y+\delta y]\,; I_7 = (-\infty,x] \cap (y+\delta y,\infty); I_8 = (x,x+\delta x] \cap (x+\delta x,\infty)$; and $I_9 = (x+\delta x,\infty) \cap (y+\delta y,\infty)$ with the corresponding probability $P_{ij} = P(\underline{W}_j \in I_i), i = 1,2,...,9$. Therefore, the joint density function $f_{k,k':n}(\underline{w})$ of $(X_{k:n}, Y_{k':n})$ is the limit of $\frac{P(E)}{\delta x \delta y}$ as $\delta x, \delta y \to 0$, where $P(E)$ can be derived by noting that $\theta_1+\theta_2+\omega = \varphi_1+\varphi_2+\omega = 1;\ r+\theta_1+s_2 = k-1;\ r+\varphi_1+s_1 = k'-1;\ r,\theta_1,s_2,\varphi_1,\omega,\theta_2,s_1,\varphi_2,t \ge 0;\ P_{1j} = F_j(\underline{w}), P_{2j} = F_j^{1,.}(\underline{w})\delta x, P_{3j} = F_{2,j}(y) - F_j(x+\delta x,y), P_{4j} = F_j^{.,1}(\underline{w})\delta y, P_{5j} \cong F_j^{1,1}(\underline{w})\delta x\delta y = f_j(\underline{w})\delta x\delta y, P_{6j} \cong (f_{2,j}(y) - F_j^{.,1}(\underline{w}+\delta\underline{w}))\delta y$, where $f_{2,j}(y) = \frac{\partial F_{2,j}(y)}{\partial y}, j = 1,2,...,n,\ \partial\underline{w} = (\delta x,\delta y), \underline{w}+\delta\underline{w} = (x+\delta x, y+\delta y), P_{7j} = F_{1,j}(x) - F_j(x,y+\delta y), P_{8j} = (f_{1,j}(x) - F_j^{1,.}(\underline{w}+\delta\underline{w}))\delta x, P_{9j} = 1 - F_{1,j}(x+\delta x) - F_{2,j}(y+\delta y) + F_j(\underline{w})$. Thus, we get

$$f_{k,k':n}(\underline{w}) = \sum_{\theta_1,\varphi_1,\theta_2,\varphi_2=0}^{1} \sum_{r=r_*}^{r^*} \sum_{\rho_{\theta_1,\theta_2,\varphi_1,\varphi_2,\omega,r}} \Pi_{j=1}^{\theta_1} P_{4i_j} \Pi_{\theta_1+1}^{\theta_1+\varphi_1} P_{2i_j} \Pi_{j=\theta_1+\varphi_1+1}^{\theta_1+\varphi_1+\theta_2} P_{6i_j} \Pi_{j=\theta_1+\varphi_1+\theta_2+1}^{\theta_1+\varphi_1+\theta_2+\varphi_2} P_{8i_j}$$
$$\Pi_{j=\theta_1+\varphi_1+\theta_2+\varphi_2+1}^{\theta_1+\varphi_1+\theta_2+\varphi_2+\omega} P_{5i_j} \Pi_{j=\theta_1+\varphi_1+\theta_2+\varphi_2+\omega+1}^{\theta_2+\varphi_1+\theta_2+\omega+k-r-1} P_{7i_j} \Pi_{j=\theta_2+\varphi_1+\varphi_2+\omega+k-r}^{\varphi_1+\theta_2+\varphi_2+\omega+k-1} P_{1i_j} \Pi_{j=\varphi_1+\theta_2+\varphi_2+\omega+k}^{\theta_2+\varphi_2+\omega+k+k'-r-2} P_{3i_j}$$
$$\Pi_{j=\theta_2+\varphi_2+\omega+k+k'-r-1}^{n} P_{9i_j}, \tag{1}$$

where $r_* = 0 \vee (k+k'+\theta_2+\varphi_2+\omega-r-1-n)$, $r^* = (k-\theta_1-1) \wedge (k'-\varphi_1-1)$, $\sum_\rho$ denotes summation subject to the condition ρ, and $\sum_{\rho_{\theta_1,\theta_2,\varphi_1,\varphi_2,\omega,r}}$ denotes the set of permutations of $i_1, ..., i_n$ such that $i_{j_1} < ... < i_{j_n}$ for each product of the type $\Pi_{j=j_1}^{j_2}$. Moreover, if $j_1 > j_2$, then $\Pi_{j=j_1}^{j_2} = 1$. But (1) can be written in the following simpler form

$$P(E) = \sum_{\theta,\varphi=0}^{1} \sum_{r=r_{**}}^{r^{**}} \sum_{\rho_{\theta,\varphi,r}} \Pi_{j=1}^{\theta} P_{4i_j} \Pi_{j=\theta+1}^{1} P_{6i_j} \Pi_{j=2}^{\varphi+1} P_{2i_j} \Pi_{j=\varphi+2}^{2} P_{8i_j} \Pi_{j=3}^{k-\theta-r+1} P_{7i_j} \Pi_{j=k-\theta-r+2}^{k-\theta+1} P_{1i_j}$$

$$\Pi_{j=k-\theta+2}^{k+k'-\theta-\varphi-r} P_{3i_j} \Pi_{j=k+k'-\theta-\varphi-r+1}^{n} P_{9i_j} + \sum_{r=0\vee(k+k'-n-1)}^{(k-1)\wedge(k'-1)} \sum_{\rho_r} P_{5i_3} \Pi_{j=2}^{k-r} P_{7i_j} \Pi_{j=k-r+1}^{k} P_{1i_j} \Pi_{j=k+1}^{k+k'-r} P_{3i_j} \Pi_{j=k+k'-r}^{n} P_{9i_j},$$

where $r_{**} = 0 \vee (k+k'-\theta-\varphi-n)$, $r^{**} = (k-\theta-1) \wedge (k'-\varphi-1)$. Therefore,

$$f_{k,k':n}(\underline{w}) = \sum_{\theta,\varphi=0}^{1} \sum_{r=r_{**}}^{r^{**}} \sum_{\rho_{\theta,\varphi,r}} \Pi_{j=1}^{\theta} P_{4i_j} \Pi_{j=\theta+1}^{1} P_{6i_j} \Pi_{j=2}^{\varphi+1} P_{2i_j} \Pi_{j=\varphi+2}^{2} P_{8i_j} \Pi_{j=3}^{k-\theta-r+1} P_{7i_j}$$

$$\Pi_{j=k-\theta-r+2}^{k-\theta+1} P_{1i_j} \Pi_{j=k-\theta+2}^{k+k'-\theta-\varphi-r} P_{3i_j} \Pi_{j=k+k'-\theta-\varphi-r+1}^{n} P_{9i_j} + \sum_{r=0\vee(k+k'-n-1)}^{(k-1)\wedge(k'-1)} \sum_{\rho_r} P_{5i_3} \Pi_{j=2}^{k-r} P_{7i_j}$$

$$\Pi_{j=k-r+1}^{k} P_{1i_j} \Pi_{j=k+1}^{k+k'-r} P_{3i_j} \Pi_{j=k+k'-r}^{n} P_{9i_j}. \tag{2}$$

Thus, we get

$$f_{k,k':n}(\underline{w}) = \sum_{\theta,\varphi=0}^{1} \sum_{r=r_{**}}^{r^{**}} \sum_{\rho_{\theta,\varphi,r}} \Pi_{j=1}^{\theta} F_{i_j}^{.,1}(\underline{w}) \Pi_{j=\theta+1}^{1} (f_{2,i_j}(y) - F_{i_j}^{.,1}(\underline{w})) \Pi_{j=2}^{\varphi+1} F_{i_j}^{1,.}(\underline{w})$$

$$\Pi_{j=\varphi+2}^{2} (f_{1,i_j}(x) - F_{i_j}^{1,.}(\underline{w})) \Pi_{j=3}^{k-\theta-r+1} (F_{2,i_j}(x) - F_{i_j}(\underline{w})) \Pi_{j=k-\theta-r+2}^{k-\theta+1} F_{i_j}(\underline{w}) \Pi_{j=k-\theta+2}^{k+k'-\theta-\varphi-r} (F_{2,i_j}(y) - F_{i_j}(\underline{w}))$$

$$\Pi_{j=k+k'-\theta-\varphi-r+1}^{n} G_{i_j}(\underline{w}) + \sum_{r=0\vee(k+k'-n-1)}^{(k-1)\wedge(k'-1)} \sum_{\rho_r} f_{i_3}(\underline{w}) \Pi_{j=2}^{k-r} (F_{1i_j}(x) - F_{i_j}(\underline{w}))$$

$$\Pi_{j=k-r+1}^{k} F_{i_j}(\underline{w}) \Pi_{j=k+1}^{k+k'-r} (F_{2,i_j}(y) - F_{i_j}(\underline{w})) \Pi_{j=k+k'-r+1}^{n} G_{i_j}(\underline{w}). \tag{3}$$

□

Hence, the proof.

Relation (3) may be written in term of permanents (c.f [9]) as follows:

$$f_{k,k':n}(\underline{w}) = \sum_{\theta,\varphi=0}^{1} \sum_{r=r_{**}}^{r^{**}} \frac{1}{(k-\theta-r-1)!\,r!\,(k'-\varphi-r-1)!\,(n-k-k'+\varphi+\theta+r-1)!}$$

$$\mathrm{Per}[\underset{\theta}{\underline{U}_{1,1}^{.,1}} \quad \underset{1-\theta}{(\underline{U}_{.,1}^{1}-\underline{U}_{1,1}^{.,1})} \quad \underset{\varphi}{\underline{U}_{1,1}^{1,.}} \quad \underset{1-\varphi}{\left(\underline{U}_{1,.}^{1}-\underline{U}_{1,1}^{1,.}\right)} \quad \underset{k-\theta-r-1}{(\underline{U}_{1,.}-\underline{U}_{1,1})} \quad \underset{r}{\underline{U}_{1,1}} \quad \underset{k'-\varphi-r-1}{(\underline{U}_{.,1}-\underline{U}_{1,1})}$$

$$\underset{n-k-k'+\theta+\varphi+r-1}{(1-\underline{U}_{1,.}-\underline{U}_{1,.}+\underline{U}_{1,1})}]$$

$$+\sum_{r=r_*}^{r^*} \frac{1}{(k-r)!\,r!\,(k'-r)!\,(n-k-k'+r)!} \mathrm{Per}[\underset{1}{\underline{U}_{1,1}^{1,1}} \quad \underset{k-r}{(\underline{U}_{1,.}-\underline{U}_{1,1})} \quad \underset{r}{\underline{U}_{1,1}} \quad \underset{k'-r}{(\underline{U}_{.,1}-\underline{U}_{1,1})} \quad \underset{n-k-k'+r-1}{(1-\underline{U}_{1,.}-\underline{U}_{1,.}+\underline{U}_{1,1})}], \tag{4}$$

where $\underline{U}_{1,.} = (F_{11}(x_1)\ \ F_{12}(x_1)\ ...\ F_{1n}(x_1))'$, $\underline{U}_{.,1} = (F_{2,1}(x_2)\ \ F_{2,2}(x_2)\ ...\ F_{2,n}(x_2))'$, $\underline{U}_{1,1} = (F_1(\underline{x})\ \ F_2(\underline{x})\ ...\ F_n(\underline{x}))'$ and $\underline{1}$ is the $n \times 1$ column vector of ones. Moreover, if $\underline{a}_1, \underline{a}_2, ...$ are column vectors, then

$$\text{Per}\begin{bmatrix} \underline{a}_1 & \underline{a}_2 & ... \\ i_1 & i_2 & ... \end{bmatrix}$$

will denote the matrix obtained by taking i_1 copies of $\underline{a}_1$, i_2 copies of $\underline{a}_2$, and so on.
Finally, when $k = k' = 1$, in (3), we get

$$f_{1,1:n}(\underline{w}) = \sum_{\rho_{\theta,\varphi,r}} (f_{2,i_1}(y) - F_{i_1}^{\cdot,1}(\underline{w}))(f_{1,i_2}(x) - F_{i_2}^{1,\cdot}(\underline{w}))\Pi_{j=3}^{n} G_{i_j}(\underline{w}) + \sum_{\rho_r} f_{i_3}(\underline{w})$$
$$(F_{2,i_2}(y) - F_{i_2}(\underline{w}))\Pi_{j=3}^{n} G_{i_3}(\underline{w}).$$

Also, for $k = k' = n$, we get

$$f_{n,n:n}(\underline{w}) = \sum_{\rho_{\theta,\varphi,r}} F_{i_1}^{\cdot,1}(\underline{w})F_{i_2}^{1,\cdot}(\underline{w})\Pi_{j=3}^{n} F_{i_j}(\underline{w}) + \sum_{\rho_r} f_{i_3}(\underline{w})\Pi_{j=2}^{n} F_{i_j}(\underline{w}))(F_{2,i_{n+1}}(y) - F_{i_{n+1}}(\underline{w})).$$

Joint distribution of the new sample rank of $X_{r:n}$ and $Y_{s:n}$

Consider n two-dimensional independent vectors $\underline{W}_j = (X_j, Y_j), j = 1, ..., n$, with the respective df $F_j(\underline{W})$ and the *jpdf* $f_j(\underline{W})$. Further assume that (X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), ..., (X_{n+m}, Y_{n+m}), $(m \geq 1)$ is another random sample with absolutely continuous df $G_j^*(x, y), j = 1, ..., m$ and *jpdf* $g_j(x, y)$. We assume that the two samples $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), ..., (X_{n+m}, Y_{n+m})$, $(m \geq 1)$ and $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are independent.

For $1 \leq r, s \leq n, m \geq 1$, we define the random variables η_1 and η_2 as follows:

$$\eta_1 = \sum_{i=1}^{m} I_{(X_{r:n} - X_{n+i})}$$

and

$$\eta_2 = \sum_{i=1}^{m} I_{(Y_{s:n} - Y_{n+i})},$$

where $I(x) = 1$ if $x > 0$ and $I(x) = 0$ if $x \leq 0$ is an indicator function. The random variables η_1 and η_2 are referred to as exceedance statistics. Clearly η_1 shows the total number of new X observations $X_{n+1}, X_{n+2}, ..., X_{n+m}$ which does not exceed a random threshold based on the rth order statistic $X_{r:n}$. Similarly, η_2 is the number of new observations $Y_{n+1}, Y_{n+2}, ..., Y_{n+m}$ which does not exceed $Y_{s:n}$.

The random variable $\zeta_1 = \eta_1 + 1$ indicates the rank of $X_{r:n}$ in the new sample $X_{n+1}, X_{n+2}, ..., X_{n+m}$, and the random variable $\zeta_2 = \eta_2 + 1$ indicates the rank of $Y_{s:n}$ in the new sample $Y_{n+1}, Y_{n+2}, ..., Y_{n+m}$. We are interested in the joint probability mass function of random variables ζ_1 and ζ_2. We will need the following representation of the compound event $P(\zeta_1 = p, \zeta_2 = q) = P(\eta_1 = p - 1, \eta_2 = q - 1)$.

Definition 1 *Denote* $A = \{X_{n+i} \leq X_{r:n}\}, A^c = \{X_{n+i} > X_{r:n}\}$, $B = \{Y_{n+i} \leq Y_{s:n}\}$ *and* $B^c = \{Y_{n+i} > Y_{s:n}\}$. *Assume that in a fourfold sampling scheme, the outcome of the random experiment is one of the events* A *or* A^c, *and simultaneously one of* B *or* B^c, *where* A^c *is the complement of* A.

In m independent repetitions of this experiment, if A appears together with B ℓ times, then A and B^c appear together $p - \ell - 1$ times. Therefore, B appears together with

A^c $q-\ell-1$ times and B^c $m-p-q+\ell+2$ times. This can be described as follows:

$A\backslash B$	B	B^c
A	ℓ	$p-\ell-1$
A^c	$q-\ell-1$	$m-p-q+\ell+2$

Clearly, the random variables η_1 and η_2 are the number of occurrences of the events A and B in m independent trials of the fourfold sampling scheme, respectively. By conditioning on $X_{r:n}=x$ and $Y_{s:n}=y$, the joint distribution of η_1 and η_2 can be obtained from bivariate binomial distribution considering the four sampling scheme with events $A=\{X_{n+i}\le x\}$, $B=\{Y_{n+i}\le y\}$, and with respective probabilities

$$\begin{aligned}
P(AB) &= P(X_{n+i}\le x, Y_{n+i}\le y),\\
P(AB^c) &= P(X_{n+i}\le x, Y_{n+i}> y),\\
P(A^cB) &= P(X_{n+i}> x, Y_{n+i}\le y),\\
P(A^cB^c) &= P(X_{n+i}> x, Y_{n+i}> y).
\end{aligned}$$

Now, we can state the following theorem.

Theorem 2 *The joint probability mass function of ζ_1 and ζ_2, is given by*

$$P(\zeta_1=p,\zeta_2=q)=P(\eta_1=p-1,\eta_2=q-1)=\sum_{\ell=max(0,p+q-m-2)}^{min(p-1,q-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$
$$\Pi_{j=1}^{\ell}G^*_{i_j}(x,y)\Pi_{j=\ell+1}^{p-1}[\,G^*_{1,i_j}(x)-G^*_{i_j}(x,y)]\,\Pi_{j=p}^{q-\ell-1+p}\left[G^*_{2,i_j}(y)-G^*_{i_j}(x,y)\right]\Pi_{j=q-\ell+p}^{m+2}\overline{G}^*_{1,i_j}(x)f_{k,k':n(\underline{w})}dxdy,$$

where, $p,q=1,...,m+1$, $f_{k,\acute{k}:n}(\underline{w})$ is defined in (3).

Proof Consider the fourfold sampling scheme described in Definition (1). By conditioning with respect to $X_{r:n}=x$ and $Y_{s:n}=y$, we obtain

$$\begin{aligned}
P(\zeta_1=p,\zeta_2=q)\equiv P(\eta_1=p-1,\eta_2=q-1) &= P\left\{\sum_{i=1}^{m}I_{(X_{r:n}-X_{n+i})}=p-1, I_{(Y_{r:n}-Y_{n+i})}=q-1\right\}\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P\left\{\sum_{i=1}^{m}I_{(X_{r:n}-X_{n+i})}=p-1, I_{(Y_{s:n}-Y_{n+i})}=q-1|X_{r:n}=x, Y_{s:n}=y\right\}\\
&\qquad\times P\{X_{r:n}=x, Y_{s:n}=y\}dxdy\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}P\left\{\sum_{i=1}^{m}I_{(x-X_{n+i})}=p-1, I_{(y-Y_{n+i})}=q-1\right\}dF_{r,s:n}(x,y).
\end{aligned}\tag{5}$$

□

On the other hand,

$$P\left(\sum_{i=1}^{m}I_{(x-X_{n+i})}=p-1, I_{(y-Y_{n+i})}=q-1\right)=\sum_{\ell=max(0,p+q-m-2)}^{min(p-1,q-1)}\Pi_{j=1}^{\ell}P_{i_j}(AB)\Pi_{j=\ell+1}^{p-1}P_{i_j}(AB^c)\,\Pi_{j=p}^{q-\ell-2+p}P_{i_j}\Pi_{j=q-\ell-1+p}^{m}P_{i_j}.\tag{6}$$

Substituting (6) in (5), we get

$$P(\zeta_1=p,\zeta_2=q)=P(\eta_1=p-1,\eta_2=q-1)=\sum_{\ell=max(0,p+q-m-2)}^{min(p-1,q-1)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Pi_{j=1}^{\ell}G^*_{i_j}(x,y)$$
$$\Pi_{j=\ell+1}^{p-1}[\,G^*_{1,i_j}(x)-G^*_{i_j}(x,y)]\,\Pi_{j=p}^{q-\ell-1+p}[\,G^*_{2,i_j}(y)-G^*_{i_j}(x,y)]\,\Pi_{j=q-\ell+p}^{m}\overline{G}^*_{1,i_j}(x)f_{k,k':n(\underline{w})}dxdy,$$

where $p,q=1,...,m+1$. This completes the proof.

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