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# Generalized mixed equilibrium problems and quasi- $\phi$ -asymptotically nonexpansive multivalued mappings in Banach spaces

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## Abstract

In this paper, we introduce two iterative algorithms for finding a common element of the set of fixed points of a quasi- $\phi$ -asymptotically nonexpansive multivalued mapping and the sets of solutions of generalized mixed equilibrium problem in Banach space. Then, we prove strong and weak convergence of the sequences to element in the mentioned set. Our results generalize and improve recent results announced by many authors.

**Keywords:** Fixed point, Generalized mixed equilibrium problem, Quasi- $\Phi$ -asymptotically nonexpansive multivalued mappings, Banach space

## Introduction

Let  $E$  be a real Banach space with norm  $\| \cdot \|$ ,  $E^*$  be the dual space of  $E$ , and  $C$  be a nonempty closed convex subset of  $E$ . Let  $f$  be a bifunction from  $C \times C$  to  $\mathbb{R}$ , where  $\mathbb{R}$  is the set of real numbers. The equilibrium problem is to find  $\hat{x} \in C$  such that

$$f(\hat{x}, y) \geq 0, \forall y \in C. \quad (1)$$

This problem was first studied by Blum and Oettli [1]. The set of solutions of equilibrium problem (1.1) is denoted by  $EP(f)$  that is  $EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \forall y \in C\}$ . Let  $A : C \rightarrow E^*$  be a nonlinear mapping. The variational inequality problem with respect to  $A$  and  $C$  is to find  $u \in C$  such that  $\langle Au, v - u \rangle \geq 0$  for all  $v \in C$ . The set of solutions of variational inequality problem with respect to  $C$  and  $A$  is denoted by  $VI(C, A)$ . Setting  $f(x, y) = \langle Ax, y - x \rangle$  for all  $x, y \in C$ , then  $\hat{x} \in EP(f)$  if and only if  $\langle A\hat{x}, y - \hat{x} \rangle \geq 0$ , for all  $y \in C$ , i.e.,  $\hat{x}$  is a solution of the variational inequality with respect to  $A$  and  $C$ . Let  $\varphi : C \rightarrow \mathbb{R} \cup \{\infty\}$  be proper, convex, and lower semi-continuous, then the minimization problem of  $\varphi$  is to find  $x \in C$  such that  $\varphi(x) \leq \varphi(y) \forall y \in C$ .

The generalized equilibrium problem is to find  $\hat{x} \in C$  such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \forall y \in C. \quad (2)$$

The set of solutions of (2) is denoted by

$$GEP(f, A) = \{\hat{x} \in C : f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle \geq 0, \forall y \in C\}.$$

In this paper, we are interested in solving equilibrium problem with respect to  $f$  given by

$$f(x, y) = \sum_{i=1}^k f_i(x, y), \forall x, y \in C, \tag{3}$$

where  $f_i : C \times C \rightarrow \mathbb{R}$  are bifunctions for  $i = 1, 2, 3, \dots, k$ , satisfying the following conditions (A<sub>1</sub>)–(A<sub>4</sub>) below;

(A<sub>1</sub>)  $f_i(x, x) = 0$ , for all  $x \in C$ , for  $i = 1, 2, 3, \dots, k$

(A<sub>2</sub>)  $f_i$  is monotone, i.e.,  $f_i(x, y) + f_i(y, x) \leq 0$ , for each  $i \in \{1, 2, 3, \dots, k\}$  and  $x, y \in C$

(A<sub>3</sub>) for all  $x, y, z \in C$ , we have  $\limsup_{t \rightarrow \infty} f_i(tz + (1 - t)x, y) \leq f_i(x, y)$

(A<sub>4</sub>) for all  $x \in C, f_i(x, \cdot)$  is convex and lower semi-continuous  $\forall i \in \{1, 2, 3, \dots, k\}$ .

The mixed equilibrium problem is to find  $\hat{x} \in C$  such that

$$\sum_{i=1}^k f_i(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C. \tag{4}$$

The set of solution of (4) is denoted by

$$GMEP(f_i, \varphi) = \left\{ \hat{x} \in C : \sum_{i=1}^k f_i(\hat{x}, y) + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C \right\}.$$

The generalized mixed equilibrium problem is to find  $\hat{x} \in C$  such that

$$\sum_{i=1}^k f_i(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C. \tag{5}$$

The set of solution of (5) is denoted by

$$GMEP(f_i, A, \varphi) = \left\{ \hat{x} \in C : \sum_{i=1}^k f_i(\hat{x}, y) + \langle A\hat{x}, y - \hat{x} \rangle + \varphi(y) - \varphi(\hat{x}) \geq 0, \forall y \in C \right\}.$$

The generalized mixed equilibrium problems are problems that arises in various applications such as in economics, mathematical physics, engineering, and other fields. Moreover, equilibrium problems are closely related with other general problems in non-linear analysis such as fixed point, game theory, variational inequality, and optimization problems. Some methods have been proposed to solve the equilibrium problem in Hilbert spaces, see for example [2–4] and references contained therein.

In 2007, Tada and Takahashi [5, 6] and Takahashi and Takahashi [7] proved weak and strong convergence theorems for finding a common element of the set of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In 2009, Takahashi and Zembayashi [8] introduced two iterative sequences for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in Banach space as follows :

$$\begin{cases} x_0 = x \in C \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = \Pi_{H_n \cap W_n} x, n \geq 0, \end{cases} \tag{6}$$

$$\begin{cases} x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \forall n \geq 0, \end{cases} \tag{7}$$

where  $J$  is the normalized duality mapping on  $E$ ,  $\{\alpha_n\} \subset [0, 1]$  satisfies  $\liminf_{n \rightarrow \infty} \alpha_n(1 - \alpha_n) > 0$  and  $\{r_n\} \subset [a, \infty]$  for some  $a > 0$ . They proved strong convergence of the scheme (6) to a common element of the set of fixed points of relatively nonexpansive mapping and the set of solution of an equilibrium problem in a Banach space. Moreover, they proved weak convergence using scheme (7). In 2012, Chang et al. [9] considered the class of uniformly quasi- $\phi$ -asymptotically nonexpansive nonself mappings and studied in a uniformly convex and uniformly smooth real Banach space. In 2014, Deng et al. [10] proved strong convergence theorems of the hybrid algorithm for common fixed point problem of finite family of asymptotically nonexpansive mappings and the set of solution of mixed equilibrium problem in uniformly smooth and uniformly convex Banach spaces. In 2016, Ezeora [11] proved strong convergence theorems for a common element of the set of solution of generalized mixed equilibrium problem and the set of common fixed points of a finite family of multivalued strictly pseudocontractive mappings in real Hilbert spaces.

In this paper, motivated and inspired by the results mentioned above, we prove strong and weak convergence theorems for finding a common element of the set of fixed point of a quasi- $\phi$ -asymptotically nonexpansive multivalued mapping and the sets of solutions of generalized mixed equilibrium problem in Banach space. Our results generalized and improve recent results announced by many authors.

**Preliminaries**

Throughout this paper, we denoted by  $\mathbb{N}$  and  $\mathbb{R}$  the sets of positive integer and real numbers, respectively. Let  $E$  be a Banach space and  $E^*$  be the dual of  $E$ ; we denote the strong convergence and the weak convergence of a sequence  $\{x_n\}$  to  $x$  in  $E$  by  $x_n \rightarrow x$  and  $x_n \rightharpoonup x$  respectively. We also denote the weak\* convergence of a sequence  $\{x_n^*\}$  to  $x^*$  in  $E^*$  by  $x_n^* \rightharpoonup x^*$ ; for all  $x \in E$  and  $x^* \in E^*$ , we denote the value of  $x^*$  at  $x$  by  $\langle x, x^* \rangle$ , which is called duality pairing. The normalized duality mapping  $J$  on  $E$  is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

for every  $x \in E$ . A Banach space  $E$  is said to be strictly convex if  $\frac{\|x + y\|}{2} < 1$  for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $x \neq y$ . The space  $E$  is also said to be uniformly convex if for each  $\varepsilon \in (0, 2]$ , there exists  $\delta > 0$  such that  $\frac{\|x + y\|}{2} \leq 1 - \delta$  for all  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $\|x - y\| \geq \varepsilon$ . A Banach space is said to have Kadec-Klee property, if for  $x_n \rightharpoonup x$  and  $\|x_n\| \rightarrow \|x\|$  imply  $x_n \rightarrow x$ . Every Hilbert space and uniformly convex Banach space has Kadec-Klee property. The space  $E$  is said to be smooth if the  $\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t}$  exists for all  $x, y \in S(E) = \{z \in E : \|z\| = 1\}$ . It is also said to be uniformly smooth if the limit exists uniformly in  $x, y \in S(E)$ . It is known that if  $E$  is smooth, strictly convex, and reflexive, then the duality mapping  $J$  is single-valued, one-to-one, and onto. The duality mapping  $J$  is said to be weakly sequentially continuous if for any sequence  $\{x_n\}$  in  $E$ ,  $x_n \rightharpoonup x$  implies  $Jx_n \rightharpoonup Jx$ , see [12].

Let  $E$  be a smooth, strictly convex, and reflexive Banach space and  $C$  be a nonempty closed convex subset of  $E$ . Throughout this paper, we denote by  $\phi$  the function defined by  $\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \forall x, y \in E$ . It is clear from the definition of the function

$\phi$  that for all  $x, y, z \in E$ , we have  $(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2$  and  $\phi(x, y) = \phi(x, z) + \phi(z, y) + \langle x - z, Jz - Jy \rangle$ . Following Albert [13], the generalized projection  $\Pi_C$  from  $E$  onto  $C$  is defined by  $\Pi_C(x) = \operatorname{argmin} \phi(y, x), \forall x \in E$  and  $y \in C$ . If  $E$  is a Hilbert space  $H$ , then  $\phi(y, x) = \|y - x\|^2$  and  $\Pi_C$  become the metric projection of  $H$  onto  $C$ .

The following lemmas for generalized projections are well known.

**Lemma 1** (see [13]) *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space  $E$ . Then,  $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \forall x \in C$  and  $y \in E$ .*

**Lemma 2** (see [13, 14]) *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space. Let  $x \in E$  and  $z \in C$ . Then,  $z = \Pi_C x \iff \langle y - z, Jx - Jy \rangle \leq 0, \forall y \in C$ .*

A mapping  $T : C \rightarrow C$  is called nonexpansive if  $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$ . We denote by  $F(T)$  the set of fixed points of  $T$  see [15]. A point  $p \in C$  is said to be an asymptotic fixed point of  $T$  if there exists a sequence  $\{x_n\}$  in  $C$  which converges weakly to  $p$  and  $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ . We denote the set of all asymptotic fixed points of  $T$  by  $\hat{F}(T)$ . Following Matsushita and Takahashi [16–18], a mapping  $T$  of  $C$  into itself is said to be relatively nonexpansive if the following conditions are satisfied:

- (i)  $F(T) \neq \emptyset$
- (ii)  $\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in F(T)$  and
- (iii)  $F(T) = \hat{F}(T)$ .

Let  $C$  be a nonempty closed convex subset of a Banach space  $E$ . Let  $\hat{CB}(C)$  be the families of nonempty, closed, and bounded subsets of  $C$

**Definition 1** *A multivalued mapping  $T : C \rightarrow \hat{CB}(C)$  is said to be relatively nonexpansive if*

- (i)  $F(T) \neq \emptyset$
- (ii)  $\phi(p, \omega) \leq \phi(p, x), \forall x \in C, \omega \in Tx, p \in F(T)$  and
- (iii)  $F(T) = \hat{F}(T)$ .

A multivalued mapping  $T : C \rightarrow \hat{CB}(C)$  is said to be closed if for any sequence  $\{x_n\} \subset C$  with  $x_n \rightarrow x$  and  $\omega_n \in T(x_n)$  with  $\omega_n \rightarrow y$  then  $y \in Tx$ .

**Definition 2** *A multivalued mapping  $T : C \rightarrow \hat{CB}(C)$  is said to be quasi- $\phi$ -nonexpansive if*

- (i)  $F(T) \neq \emptyset$  and
- (ii)  $\phi(p, \omega) \leq \phi(p, x), \forall x \in C, \omega \in Tx, p \in F(T)$ .

A multivalued mapping  $T : C \rightarrow \hat{CB}(C)$  is said to be quasi- $\phi$ -asymptotically nonexpansive if

- (i)  $F(T) \neq \emptyset$
- (ii) There exists a real sequence  $\{k_n\} \subset [1, \infty)$  with  $k_n \rightarrow 1$  such that  $\phi(p, \omega_n) \leq k_n \phi(p, x), \forall n \geq 1, x \in C, \omega_n \in T^n x, p \in F(T)$ .

**Lemma 3** (see [14]) *Let  $E$  be a smooth and uniformly convex Banach space and let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $E$  such that either  $\{x_n\}$  or  $\{y_n\}$  is bounded. If  $\lim_{n \rightarrow \infty} \phi(x_n, y_n) = 0$ , then  $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ .*

**Lemma 4** (see [19]) *Let  $E$  be a uniformly convex Banach space. For arbitrary  $r > 0$ , let  $B_r(0) := \{x \in E : \|x\| \leq r\}$ . Then, for any given sequence  $\{x_n\}_{n=1}^\infty \subset B_r(0)$  and for any given sequence  $\{\lambda_n\}_{n=1}^\infty$  of positive numbers such that  $\sum_{n=1}^\infty \lambda_n = 1$ , there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R}, g(0) = 0$$

such that for any positive integers  $i, j$  with  $i < j$ , the following inequality holds:

$$\left\| \sum_{n=1}^\infty \lambda_n x_n \right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|).$$

**Lemma 5** (see [20]) *Let  $E$  be a smooth and uniformly convex Banach space and let  $r > 0$ . Then, there exists a strictly increasing continuous and convex function  $g : [0, 2r] \longrightarrow \mathbb{R}$  such that  $g(0) = 0$  and  $g_1(\|x - y\|) \leq \phi(x, y)$ , for all  $x, y \in B_r$ .*

**Lemma 6** (see [21]) *Let  $\{a_n\}$ ,  $\{b_n\}$ , and  $\{c_n\}$  be sequences of nonnegative real numbers satisfying  $a_{n+1} \leq (1 + c_n)a_n + b_n$ , for all  $n \in \mathbb{N}$ , where  $\sum_{n=1}^\infty b_n < \infty$  and  $\sum_{n=1}^\infty c_n < \infty$ . Then,*

- (i)  $\lim_{n \rightarrow \infty} a_n$  exists.
- (ii) if  $\liminf_{n \rightarrow \infty} a_n = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 7** (see [22]) *Let  $E$  be a smooth, strictly convex, and reflexive Banach space and  $C$  be a nonempty closed convex subset of  $E$ . Let  $B : C \longrightarrow E^*$  be a continuous and monotone mapping,  $\zeta : C \longrightarrow \mathbb{R}$  be a lower semi-continuous and convex function, and  $h : C \times C \longrightarrow \mathbb{R}$  be a bifunction satisfying the conditions (A1) – (A4). Let  $r > 0$  be any given number and  $u \in E$  be any given point. Then, the following hold:*

- (1) There exists  $z \in C$  such that

$$h(z, v) + \zeta(v) - \zeta(z) + \langle v - z, Bz \rangle + \frac{1}{r} \langle v - z, Jz - Ju \rangle \geq 0, \forall v \in C.$$

- (2) If we define a mapping  $A_r : E \longrightarrow C$  by

$$A_r(u) = \left\{ z \in C : h(z, v) + \zeta(v) - \zeta(z) + \langle v - z, Bz \rangle + \frac{1}{r} \langle v - z, Jz - Ju \rangle \geq 0, \forall v \in C \right\}, u \in E,$$

the mapping  $A_r$  has the following properties:

- (a)  $A_r$  is single-valued;
  - (b)  $F(A_r) = \text{GMEP}(h, A, \zeta) = \hat{F}(A_r)$
  - (c)  $\text{GMEP}(h, A, \zeta)$  is a closed convex subset of  $C$ ;
  - (d)  $\phi(q, A_r u) + \phi(A_r u, u) \leq \phi(q, u), \forall q \in F(A_r), u \in E$ .
- where  $\hat{F}(A_r)$  denotes the set of asymptotic fixed points of  $A_r$ , i.e.,

$$\hat{F}(A_r) := \{x \in C : \exists \{x_n\} \subset C, \text{ s.t. } x_n \rightharpoonup x, \|x_n - A_r x_n\| \longrightarrow 0 (n \longrightarrow \infty)\}$$

### Strong convergence theorem

In this section, we prove a strong convergence theorem for finding a common element of

the set of solutions of generalized mixed equilibrium problems and the set of fixed point of quasi- $\phi$ -asymptotically nonexpansive multivalued mapping in Banach space.

**Theorem 1** *Let  $E$  be a uniformly smooth and uniformly convex Banach space, and Let  $C$  be a nonempty closed convex subset of  $E$  and  $\hat{CB}(C)$  be the family of nonempty, closed, and bounded subsets of  $C$ . Let  $f_i : C \times C \rightarrow \mathbb{R}, i = 1, 2, 3, \dots, k$  be bi functions which satisfy the conditions  $(A_1) - (A_4)$ ,  $A : C \rightarrow E^*$  be a nonlinear mapping, and  $\varphi : C \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper, convex, and lower semi-continuous function. Let  $T_i, i = 1, 2, 3, \dots, N$  be a quasi- $\phi$ -asymptotically nonexpansive multivalued mapping from  $C$  into  $\hat{CB}(C)$  such that  $F(T) \cap GMEP(f, A, \varphi) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by*

$$\begin{cases} x_0 = x \in C \\ y_n = J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right), \\ u_n \in C \text{ such that } \sum_{i=1}^K f_i(u_n, y) + \varphi(y) - \varphi(u_n) + \langle y - u_n, Au_n \rangle \\ + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ M_n = \{z \in C : \phi(z, u_n) \leq k_n^2 \phi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = \Pi_{M_n \cap W_n} x, \forall n \geq 0, \end{cases} \tag{8}$$

where  $J$  is the normalized duality mapping of  $E$ ,  $\{\alpha_{i,n}\} \subset [0, 1]$  satisfies  $\liminf_{n \rightarrow \infty} \alpha_{0,n} > 0$ ,  $\sum_{i=0}^N \alpha_{i,n} = 1$  and  $w_{i,n} \in T_i^n x_n, \forall i = 1, 2, 3, \dots, N$ .  $\{r_n\} \subset [a, \infty]$ , some  $a > 0$ . Then,  $\{x_n\}$  converges strongly to  $\Pi_{F(T) \cap GMEP(f, A, \varphi)} x$ , where  $\Pi_{F(T) \cap GMEP(f, A, \varphi)}$  is the generalized projection of  $E$  onto  $F(T) \cap GMEP(f, A, \varphi)$ ,

*Proof* Let two functions  $\tau : C \times C \rightarrow \mathbb{R}$  and  $T_r : E \rightarrow C$  be defined by

$$\tau(x, y) = \sum_{i=1}^k f_i(x, y) + \langle Ax, y - x \rangle + \varphi(y) - \varphi(x), \forall x, y \in C$$

and

$$T_r(x) = \left\{ u \in C : \tau(u, y) + \frac{1}{r} \langle y - u, Ju - Jx \rangle \geq 0, \forall y \in C, \right\} \forall x \in E$$

respectively. Now, the function  $\tau$  satisfies conditions  $(A1) - (A4)$  and  $T_r$  has the properties  $(a) - (d)$ . Therefore, iterative sequence (8) can be rewritten as

$$\begin{cases} x_0 = x \in C \\ y_n = J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right), w_{i,n} \in T_i^n x_n, n \geq 1 \\ u_n \in C \text{ such that } \tau(u_n, y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jy_n \rangle \geq 0, \forall y \in C \\ M_n = \{z \in C : \phi(z, u_n) \leq k_n^2 \phi(z, x_n)\} \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\} \\ x_{n+1} = \Pi_{M_n \cap W_n} x, N \in \mathbb{N}. \end{cases} \tag{9}$$

□

We first show that  $M_n \cap W_n$  is closed and convex, and it is obvious that  $M_n$  is closed and convex since  $\phi(z, u_n) \leq k_n^2 \phi(z, x_n) \iff (1 - k_n^2) \|z\|^2 - 2(1 - k_n^2) \langle z, Ju_n \rangle + 2k_n^2 \langle z, Jx_n - Ju_n \rangle \leq k_n^2 \|x_n\|^2 - \|u_n\|^2$ .

Thus,  $M_n \cap W_n$  is a closed and convex subset of  $E$  for all  $n \in \mathbb{N} \cup \{0\}$ , so that  $\{x_n\}$  is well defined.

Let  $u \in F(T) \cap GMEP(f, A, \varphi)$ , putting  $u_n = \omega_n \in T_{r_n}y_n$  for all  $n \in \mathbb{N} \cup \{0\}$ , and since  $T_{r_n}$  are quasi- $\phi$ -asymptotically nonexpansive multivalued, we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, \omega_n) \\ &\leq k_n \phi(u, y_n) \\ &= k_n \left[ \phi \left( u, J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right) \right) \right], \\ &= k_n \left[ \|u\|^2 - 2 \left\langle u, \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle + \left\| \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\|^2 \right] \\ &\leq k_n \left[ \|u\|^2 - 2 \left( \langle u, \alpha_{0,n} Jx_n \rangle + \left\langle u, \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle \right) + \alpha_{0,n} \|Jx_n\|^2 + \sum_{i=1}^N \alpha_{i,n} \|Jw_{i,n}\|^2 \right] \\ &= k_n \left[ \alpha_{0,n} (\|u\|^2 - 2 \langle u, Jx_n \rangle + \|x_n\|^2) + \sum_{i=1}^N \alpha_{i,n} (\|u\|^2 - 2 \langle u, Jw_{i,n} \rangle + \|w_{i,n}\|^2) \right] \\ &= k_n \left[ \alpha_{0,n} \phi(u, x_n) + \sum_{i=1}^N \alpha_{i,n} \phi(u, w_{i,n}) \right] \\ &\leq k_n \left[ \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) \right] \\ &\leq k_n \left[ k_n \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) \right] \\ &= k_n \left[ \left( \alpha_{0,n} + \sum_{i=1}^N \alpha_{i,n} \right) k_n \phi(u, x_n) \right] \\ &= k_n^2 \phi(u, x_n). \end{aligned}$$

Hence, we have  $u \in M_n$ . This implies that  $F(T) \cap GMEP(f, A, \varphi) \subset M_n, \forall n \in \mathbb{N} \cup \{0\}$ .

Next, we show by induction that  $F(T) \cap GMEP(f, A, \varphi) \subset M_n \cap W_n, \forall n \in \mathbb{N} \cup \{0\}$ .

From  $W_0 = C$ , we have

$$F(T) \cap GMEP(f, A, \varphi) \subset M_0 \cap W_0, .$$

Suppose that  $F(T) \cap GMEP(f, A, \varphi) \subset M_k \cap W_k$ , for some  $k \in \mathbb{N} \cup \{0\}$ . Then, there exists  $x_{k+1} \in M_k \cap W_k$  such that  $x_{k+1} = \Pi_{M_k \cap W_k} x$

From the definition of  $x_{k+1}$ , we have for all  $z \in M_k \cap W_k$ ,

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0.$$

Since  $F(T) \cap GMEP(f, A, \varphi) \subset M_k \cap W_k$ , we have

$$\langle x_{k+1} - z, Jx - Jx_{k+1} \rangle \geq 0.$$

$\forall z \in F(T) \cap GMEP(f, A, \varphi)$  and so  $z \in W_{k+1}$ . Thus,  $F(T) \cap GMEP(f, A, \varphi) \subset W_{k+1}$ . Therefore, we have  $F(T) \cap GMEP(f, A, \varphi) \subset M_{k+1} \cap W_{k+1}$ . Therefore, we obtain

$$F(T) \cap GMEP(f, A, \varphi) \subset M_n \cap W_n, \forall n \in \mathbb{N} \cup \{0\}.$$

From the definition of  $W_n$ , we have  $x_n = \Pi_{W_n}x$ ; using this and Lemma 1, we have

$$\begin{aligned} \phi(x_n, x) &= \phi(\Pi_{W_n}x, x) \leq \phi(u, x) - \phi(u, \Pi_{W_n}x) \\ &\leq \phi(u, x) \end{aligned} \tag{10}$$

for all  $u \in F(T) \cap GMEP(f, A, \varphi) \subset W_n$ . Therefore,  $\phi(x_n, x)$  is bounded, and consequently  $\{x_n\}$  and  $\{T_i^n x_n\}$  are bounded.

Since  $x_{n+1} = \Pi_{M_n \cap W_n}x \in M_n \cap W_n \subset W_n$  and  $x_n = \Pi_{W_n}x$ , we have

$$\phi(x_n, x) \leq \phi(x_{n+1}, x), \forall n \in \mathbb{N} \cup \{0\}. \tag{11}$$

Thus,  $\{\phi(x_n, x)\}$  is nondecreasing. Using (10) and (11), we have the limit of  $\{\phi(x_n, x)\}$  exists.

From  $x_n = \Pi_{W_n}x$  and Lemma 1, we also have

$$\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{W_n}x) \leq \phi(x_{n+1}, x) - \phi(\Pi_{W_n}x, x) = \phi(x_{n+1}, x) - \phi(x_n, x)$$

for all  $n \in \mathbb{N} \cup \{0\}$ . This means that  $\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0$ .

From  $x_{n+1} = \Pi_{M_n \cap W_n}x \in M_n$  and the definition of  $M_n$ , we have

$$\phi(x_{n+1}, u_n) \leq k_n^2 \phi(x_{n+1}, x_n), \forall n \in \mathbb{N} \cup \{0\}.$$

Therefore, we have

$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0$ . As  $E$  is uniformly convex and smooth, we have from Lemma 3 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0.$$

From which, we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0.$$

Since  $J$  is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

Let  $r = \sup_{n \in \mathbb{N}} \{\|x_n\|, \|T_i^n x_n\|\}$ . Since  $E$  is a uniformly smooth Banach space, we know that  $E^*$  is a uniformly convex Banach space. So, for  $u \in F(T) \cap GMEP(f, A, \varphi)$ , putting  $u_n = \omega_n = T_{r_n}y_n$  and using Lemma 4, we have :

$$\begin{aligned}
 \phi(u, u_n) &= \phi(u, \omega_n) \\
 &\leq k_n \phi(u, y_n) \\
 &= k_n \left[ \phi \left( u, J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right) \right) \right], \\
 &= k_n \left[ \|u\|_2 - 2 \left\langle u, \alpha_{i,0} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle + \left\| \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\|^2 \right] \\
 &\leq k_n \left[ \|u\|^2 - 2(\langle u, \alpha_{0,n} Jx_n \rangle + \langle u, \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \rangle) + \alpha_{0,n} \|Jx_n\|^2 + \sum_{i=1}^N \alpha_{i,n} \|Jw_{i,n}\|^2 \right. \\
 &\quad \left. - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n \left[ \alpha_{0,n} [\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2] + \sum_{i=1}^N \alpha_{i,n} [\|u\|^2 - 2\langle u, Jw_{i,n} \rangle + \|w_{i,n}\|^2] \right. \\
 &\quad \left. - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n \left[ \alpha_{0,n} \phi(u, x_n) + \sum_{i=1}^N \alpha_{i,n} \phi(u, w_{i,n}) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &\leq k_n \left[ \alpha_{0,n} \phi(u, x_n) + \sum_{i=1}^N k_n \alpha_{i,n} \phi(u, x_n) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &\leq k_n \left[ k_n \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n [k_n \phi(u, x_n) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|)] \\
 &= k_n^2 \phi(u, x_n) - k_n \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|). \tag{12}
 \end{aligned}$$

Therefore, from (12), we have

$$k_n \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \leq k_n^2 \phi(u, x_n) - \phi(u, u_n), \forall n \in \mathbb{N} \cup \{0\}.$$

But

$$\begin{aligned}
 k_n^2 \phi(u, x_n) - \phi(u, u_n) &= k_n^2 [\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2] - [\|u\|^2 - 2\langle u, Ju_n \rangle + \|u_n\|^2] \\
 &= (k_n^2 - 1) \|u\|^2 - 2(k_n^2 - 1) \langle u, Ju_n \rangle - 2k_n^2 \langle u, Jx_n - Ju_n \rangle + k_n^2 \|x_n\|^2 - \|u_n\|^2 \\
 &= (k_n^2 - 1) \|u\|^2 - 2(k_n^2 - 1) \langle u, Ju_n \rangle - 2k_n^2 \langle u, Jx_n - Ju_n \rangle + (k_n^2 - 1) \|x_n\|^2 \\
 &\quad + \|x_n\|^2 - \|u_n\|^2 \\
 &\leq |(k_n^2 - 1) \|u\|^2| + |2(k_n^2 - 1) \langle u, Ju_n \rangle| + |2k_n^2 \langle u, Jx_n - Ju_n \rangle| + |(k_n^2 - 1) \|x_n\|^2| \\
 &\quad + |\|x_n\|^2 - \|u_n\|^2| \\
 &\leq (k_n^2 - 1) \|u\|^2 + 2(k_n^2 - 1) \|u\| \|Ju_n\| + 2k_n^2 \|u\| \|Jx_n - Ju_n\| + (k_n^2 - 1) \|x_n\|^2 \\
 &\quad + (\|x_n - u_n\|)(\|x_n\| + \|u_n\|)
 \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} (k_n^2 \phi(u, x_n) - \phi(u, u_n)) = 0. \tag{13}$$

Since  $\liminf_{n \rightarrow \infty} \alpha_{0,n} \alpha_{i,n} > 0$ , we have

$$\lim_{n \rightarrow \infty} g(\|Jx_n - Jw_{i,n}\|) = 0.$$

From the property  $g$ , we have  $\lim_{n \rightarrow \infty} \|Jx_n - Jw_{i,n}\| = 0$ .

Since  $J^{-1}$  is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - w_{i,n}\| = 0.$$

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x_{n_k} \rightarrow \hat{x}$ , for some  $\hat{x} \in E$ . Since  $T$  is quasi- $\phi$ -asymptotically nonexpansive multivalued mapping and  $E$  is a reflexive space, then we have  $\hat{x} \in F(T_i)$ .

Next, we show that  $\hat{x} \in GMEP(f, A, \varphi)$ . From  $u_n = T_{r_n}y_n$  Lemma 7(d) and (13), we have

$$\begin{aligned} \phi(u_n, y_n) &= \phi(T_{r_n}y_n, y_n) \\ &\leq \phi(u, y_n) - \phi(u, T_{r_n}y_n) \\ &\leq k_n^2(u, x_n) - \phi(u, T_{r_n}y_n) \\ &= k_n^2\phi(u, x_n) - \phi(u, u_n). \end{aligned}$$

Thus,  $\lim_{n \rightarrow \infty} \phi(u_n, y_n) = 0$ .

Since  $E$  is uniformly convex and smooth, we have from Lemma 3 that

$$\lim_{n \rightarrow \infty} \|u_n - y_n\| = 0. \tag{14}$$

From  $x_{n_k} \rightarrow \hat{x}$ ,  $\|x_n - u_n\| \rightarrow 0$  and (14), we have  $y_{n_k} \rightarrow \hat{x}$  and  $u_{n_k} \rightarrow \hat{x}$ .

As  $J$  is uniformly norm-to-norm continuous on bounded sets and (14), we have

$\lim_{n \rightarrow \infty} \|Ju_n - Jy_n\| = 0$ . From  $r_n \geq a$ , we have

$$\lim_{n \rightarrow \infty} \left\| \frac{Ju_n - Jy_n}{r_n} \right\| = 0. \tag{15}$$

By  $u_n = T_{r_n}y_n$ , we have

$$\tau(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jy_n) \geq 0, \forall y \in C.$$

Replacing  $n$  by  $n_k$ , we have from (A<sub>2</sub>) that

$$\frac{1}{r_{n_k}}(y - u_{n_k}, Ju_{n_k} - Jy_{n_k}) \geq -\tau(u_{n_k}, y) \geq \tau(y, u_{n_k}), \forall y \in C \tag{16}$$

Letting  $k \rightarrow \infty$ , in (16) and using (A<sub>4</sub>), we obtain

$$\tau(y, \hat{x}) \leq 0, \forall y \in C.$$

For  $t$  with  $0 < t \leq 1$  and  $y \in C$ , let  $y_t = ty + (1 - t)\hat{x}$ . Since  $y \in C$  and  $\hat{x} \in C$ , we have  $y_t \in C$  and  $\tau(y_t, \hat{x}) \leq 0, \forall y \in C$ . Now, using (A<sub>1</sub>) and (A<sub>3</sub>), we have

$$\begin{aligned} 0 &= \tau(y_t, y_t) \\ &\leq t\tau(y_t, y) + (1 - t)\tau(y_t, \hat{x}) \\ &\leq t\tau(y_t, y). \end{aligned}$$

Dividing by,  $t$  we have

$$\tau(y_t, y) \geq 0, \forall y \in C.$$

Letting  $t \rightarrow 0$ , and using  $(A_3)$ , we have

$$\tau(\hat{x}, y) \geq 0, \forall y \in C.$$

This shows that  $\hat{x} \in GMEP(f, A, \varphi)$ .

Let  $\omega = \Pi_{F(T) \cap GMEP(f, A, \varphi)} x$ , From  $x_{n+1} = \Pi_{M_n \cap W_n} x$  and  $\omega \in F(T) \cap GMEP(f, A, \varphi) \subset M_n \cap W_n$ , we have

$$\phi(x_{n+1}, x) \leq \phi(\omega, x).$$

Since the norm is weakly lower semi-continuous and  $x_{n_k} \rightharpoonup \hat{x}$ , we have

$$\begin{aligned} \phi(\hat{x}, x) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx \rangle + \|x\|^2 \\ &\leq \liminf_{k \rightarrow \infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx \rangle + \|x\|^2) \\ &= \liminf_{k \rightarrow \infty} \phi(x_{n_k}, x) \\ &\leq \limsup_{k \rightarrow \infty} \phi(x_{n_k}, x) \\ &\leq \phi(\omega, x). \end{aligned}$$

From the definition of  $\Pi_{F(T) \cap GMEP(f, A, \varphi)}$ , we have  $\hat{x} = \omega$ . Hence,  $\lim_{k \rightarrow \infty} \phi(x_{n_k}, x) = \phi(\omega, x)$ , Therefore,

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (\phi(x_{n_k}, x) - \phi(\omega, x)) \\ &= \lim_{k \rightarrow \infty} (\|x_{n_k}\|^2 - \|\omega\|^2 - 2\langle x_{n_k} - \omega, Jx \rangle) \\ &= \lim_{k \rightarrow \infty} (\|x_{n_k}\|^2 - \|\omega\|^2). \end{aligned}$$

Since  $E$  has the Kadec-Klee property, we have that  $x_{n_k} \rightarrow \omega = \Pi_{F(T) \cap GMEP(f, A, \varphi)} x$ . Therefore,  $\{x_n\}$  converges strongly to  $\Pi_{F(T) \cap GMEP(f, A, \varphi)} x$ .

### Weak convergence theorem

In this section, we prove a weak convergence theorem for finding a common element of the set of solutions of generalized mixed equilibrium problem and the set of fixed point of quasi- $\phi$ -asymptotically nonexpansive multivalued mapping in Banach space. Before proving the Theorem, we need the following proposition.

**Proposition 1** *Let  $E$  be a uniformly smooth and uniformly convex Banach space and let  $C$  be a nonempty closed convex subset of  $E$  and  $\hat{CB}(C)$  be the family of nonempty, closed, and bounded subsets of  $C$ . Let  $f_i : C \times C \rightarrow \mathbb{R}, (i = 1, 2, 3, \dots, K \in \mathbb{N})$  be bifunctions satisfying  $(A_1) - (A_4)$ ,  $A : C \rightarrow E^*$  be a nonlinear mapping and let  $\varphi : C \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper, convex, and lower semi-continuous function. Let  $T$  be a quasi- $\phi$ -asymptotically nonexpansive multivalued mapping from  $C$  into  $\hat{CB}(C)$  such that  $F(T) \cap GMEP(f, A, \varphi) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by  $u_1 \in E$*

$$\begin{cases} x_n \in C \text{ such that } \tau(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C \\ u_{n+1} = J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N Jw_{i,n} \right), \quad w_{i,n} \in T_i^n x_n, K, N, \in \mathbb{N} \end{cases}$$

for every  $n \in \mathbb{N}$ , where  $J$  is the normalized duality mapping on  $E$ ,  $\{\alpha_{i,n}\} \subset [0, \infty)$  satisfying  $\liminf_{n \rightarrow \infty} \alpha_{0,n} > 0, \sum_{i=0}^N \alpha_{i,n} = 1$  and  $w_{i,n} \in T_i^n x_n, \forall i = 1, 2, 3, \dots, N \in \mathbb{N}$ .

Let  $\{r_n\} \subset (0, \infty)$ . Then,  $\left\{ \Pi_{F(T) \cap GMEP(f,A,\varphi)} x_n \right\}$  converges strongly to  $z \in F(T) \cap GMEP(f,A,\varphi)$ , where  $\Pi_{F(T) \cap GMEP(f,A,\varphi)}$  is generalized projection of  $E$  onto  $F(T) \cap GMEP(f,A,\varphi)$ .

*Proof* Let  $u \in F(T) \cap GMEP(f,A,\varphi)$ . Putting  $x_n = \omega_n \in T_{r_n} u_n$  for all  $n \in \mathbb{N}$ , we know that  $T_{r_n}$  are quasi- $\phi$ -asymptotically nonexpansive multivalued, and we have

$$\begin{aligned} & \phi(u, x_{n+1}) \\ &= \phi(u, \omega_{n+1}) \\ &\leq k_n \phi(u, u_{n+1}) \\ &= k_n \left[ \phi(u, J^{-1}(\alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n})) \right], \quad w_{i,n} \in T_i^n x_n, n \geq 1 \\ &= k_n \left[ \|u\|^2 - 2 \left\langle u, \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle + \left\| \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\|^2 \right] \\ &\leq k_n \left[ \|u\|^2 - 2 \left( \langle u, \alpha_{0,n} Jx_n \rangle + \left\langle u, \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle \right) + \alpha_{0,n} \|Jx_n\|^2 + \sum_{i=1}^N \alpha_{i,n} \|Jw_{i,n}\|^2 \right] \\ &= k_n \left[ \alpha_{0,n} (\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2) + \sum_{i=1}^N \alpha_{i,n} (\|u\|^2 - 2\langle u, Jw_{i,n} \rangle + \|w_{i,n}\|^2) \right] \\ &\leq k_n \left[ \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) \right] \\ &\leq k_n \left[ k_n \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) \right] \\ &= k_n^2 \phi(u, x_n). \end{aligned}$$

Thus,

$$\phi(u, x_{n+1}) \leq k_n^2 \phi(u, x_n) \tag{17}$$

Hence, we have

$$\phi(u, x_{n+1}) \leq k_n^2 \phi(u, x_n) = (1 + (k_n^2 - 1)) \phi(u, x_n)$$

By Lemma 6,  $\sum_{i=1}^\infty (k_n^2 - 1) < \infty$ , we obtain  $\lim_{n \rightarrow \infty} \phi(u, x_n)$  exists. It follows that  $\{x_n\}$  and  $\{w_{i,n}\}$  are bounded.

Define  $y_n = \Pi_{F(T) \cap GMEP(f,A,\varphi)} x_n$  for all  $n \in \mathbb{N}$ . Then,  $y_n \in F(T) \cap GMEP(f,A,\varphi)$ ; therefore, from (17), we have

$$\phi(y_n, x_{n+1}) \leq k_n^2 \phi(y_n, x_n). \tag{18}$$

Thus,

$$\begin{aligned} \phi(y_{n+1}, x_{n+1}) &= \phi\left(\Pi_{F(T) \cap GMEP(f,A,\varphi)} x_{n+1}, x_{n+1}\right) \\ &\leq \phi(y_n, x_{n+1}) - \phi\left(y_n, \Pi_{F(T) \cap GMEP(f,A,\varphi)} x_{n+1}\right) \\ &= \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1}) \\ &\leq \phi(y_n, x_{n+1}). \end{aligned}$$

Hence, from (18), we have  $\phi(y_{n+1}, x_{n+1}) \leq k_n^2 \phi(y_n, x_n) = (1 + (k_n^2 - 1)) \phi(y_n, x_n)$ .

By the assumption  $\sum_{i=1}^\infty (k_n^2 - 1) < \infty$  and using Lemma 6, we have  $\lim_{n \rightarrow \infty} \phi(y_n, x_n)$ . For  $m \in \mathbb{N}$  such that  $m > n$ , we also have from (18) that

$$\phi(y_n, x_{n+m}) \leq (k_n^2)^m \phi(y_n, x_n).$$

From  $y_{n+m} = \Pi_{F(T) \cap GMEP(f, A, \varphi)} x_{n+m}$  and Lemma 1, we have

$$\begin{aligned} \phi(y_n, y_{n+m}) + \phi(y_{n+m}, x_{n+m}) &\leq \phi(y_n, x_{n+m}) \\ &\leq (k_n^2)^m \phi(y_n, x_n) \end{aligned}$$

Hence,

$\phi(y_n, y_{n+m}) \leq (k_n^2)^m \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m})$ . Let  $r = \sup_{n \in \mathbb{N}} \|y_n\|$ ; from Lemma 5, we have

$$g(\|y_n - y_{n+m}\|) \leq \phi(y_n, y_{n+m}) \leq (k_n^2)^m \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m})$$

Since  $\lim_{n \rightarrow \infty} \phi(y_n, x_n)$  exists, from the property of  $g$ , we have that  $\{y_n\}$  is Cauchy. Since  $F(T) \cap GMEP(f, A, \varphi)$  is closed,  $\{y_n\}$  converges strongly to  $z \in F(T) \cap GMEP(f, A, \varphi)$ .  $\square$

Now, we prove the following theorem.

**Theorem 2** *Let  $E$  be a real uniformly smooth and uniformly convex Banach space, let  $C$  be a nonempty, closed, and convex subset of  $E$ , and let  $\hat{C}B(C)$  be family of nonempty, closed, and bounded subsets of  $C$ . Let  $f_i : C \times C \rightarrow \mathbb{R}$ , ( $i = 1, 2, 3, \dots, K$ ) be bifunctions satisfying  $(A_1) - (A_4)$ ,  $A : C \rightarrow E^*$  be a nonlinear mapping, and  $\varphi : C \rightarrow \mathbb{R} \cup \{\infty\}$  be a proper, convex, and lower semi-continuous function. Let  $T$  be a quasi- $\phi$ -asymptotically nonexpansive multivalued mapping from  $C$  into  $\hat{C}B(C)$  such that  $F(T) \cap GMEP(f, A, \varphi) \neq \emptyset$ . Let  $\{x_n\}$  be a sequence generated by  $u_1 \in E$*

$$\begin{cases} x_n \in C \text{ such that } \tau(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C. \\ u_{n+1} = J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right), \quad w_{i,n} \in T_i^n x_n, K, N, \in \mathbb{N} \end{cases}$$

for every  $n \in \mathbb{N}$ , where  $J$  is the normalized duality mapping of  $E$ ,  $\{\alpha_{i,n}\} \subset [0, \infty]$  satisfies  $\liminf_{n \rightarrow \infty} \alpha_{0,n} > 0$ ,  $\sum_{i=0}^N \alpha_{i,n} = 1$  and  $w_{i,n} \in T_i^n x_n, \forall i = 1, 2, 3, \dots, N \in \mathbb{N}$ .

Let  $\{r_n\} \subset [a, \infty)$  for some  $a > 0$ . If  $J$  is weakly sequentially continuous, then  $\{x_n\}$  converges weakly to  $z \in F(T) \cap GMEP(f, A, \varphi)$ , where  $z = \lim_{n \rightarrow \infty} \Pi_{F(T) \cap GMEP(f, A, \varphi)} x_n$ .

*Proof* As in the proof of proposition 17, we have that  $\{x_n\}$  and  $\{w_{i,n}\}$  and  $\{T_i^n x_n\}$  are bounded sequences. Let  $r = \sup_{n \in \mathbb{N}} \{\|x_n\|, \|w_{i,n}\|\}$ . For  $u \in F(T) \cap GMEP(f, A, \varphi)$ , we have

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, \omega_{n+1}) \\ &\leq k_n \phi(u, u_{n+1}) \\ &= k_n \left[ \phi \left( u, J^{-1} \left( \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right) \right) \right] \\ &= k_n \left[ \|u\|^2 - 2 \left\langle u, \alpha_{i,0} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle + \left\| \alpha_{0,n} Jx_n + \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\|^2 \right] \end{aligned}$$

$$\begin{aligned}
 &\leq k_n \left[ \|u\|^2 - 2(\langle u, \alpha_{0,n} Jx_n \rangle + \left\langle u, \sum_{i=1}^N \alpha_{i,n} Jw_{i,n} \right\rangle) \right. \\
 &\quad \left. + \alpha_{0,n} \|x_n\|^2 + \sum_{i=1}^N \alpha_{i,n} \|w_{i,n}\|^2 - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n \left[ \alpha_{0,n} (\|u\|^2 - 2\langle u, Jx_n \rangle + \|x_n\|^2) + \sum_{i=1}^N \alpha_{i,n} (\|u\|^2 - 2\langle u, Jw_{i,n} \rangle + \|w_{i,n}\|^2) \right. \\
 &\quad \left. - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n \left[ \alpha_{0,n} \phi(u, x_n) + \sum_{i=1}^N \alpha_{i,n} \phi(u, w_{i,n}) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &\leq k_n \left[ k_n \alpha_{0,n} \phi(u, x_n) + k_n \sum_{i=1}^N \alpha_{i,n} \phi(u, x_n) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \right] \\
 &= k_n [k_n \phi(u, x_n) - \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|)] \\
 &= k_n^2 \phi(u, x_n) - k_n \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|).
 \end{aligned}$$

Thus,

$$\phi(u, x_{n+1}) \leq k_n^2 \phi(u, x_n) - k_n \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|). \tag{19}$$

Hence, we have

$$k_n \alpha_{0,n} \alpha_{i,n} g(\|Jx_n - Jw_{i,n}\|) \leq k_n^2 \phi(u, x_n) - \phi(u, x_{n+1}).$$

Since  $\{\phi(u, x_n)\}$  is convergent, we have

$$\lim_{n \rightarrow \infty} (k_n^2 \phi(u, x_n) - \phi(u, x_{n+1})) = 0.$$

As, we get  $\lim_{n \rightarrow \infty} \alpha_{0,n} \alpha_{i,n} > 0$

$$\lim_{n \rightarrow \infty} g(\|Jx_n - Jw_{i,n}\|) = 0.$$

From the property of  $g$ , we have

$$\lim_{n \rightarrow \infty} \|Jx_n - Jw_{i,n}\| = 0.$$

Since  $J^{-1}$  is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|x_n - w_{i,n}\| = 0. \tag{20}$$

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\{x_{n_k}\}$  converges weakly to  $\hat{x} \in C$ . From (20) and  $F(T)$ , we have  $\hat{x} \in F(T)$ .

Next, we show that  $\hat{x} \in GMEP(f, A, \varphi)$ . Let  $T = \sup_{n \in \mathbb{N}} \{\|x_n\|, \|u_n\|\}$  from Lemma 5, and putting  $x_n = T_{r_n} u_n$ , we have from Lemma 7(d) and (19) that for  $u \in F(T) \cap GMEP(f, A, \varphi)$ .

$$\begin{aligned}
 g_1(\|x_n - u_n\|) &\leq \phi(x_n, u_n) \\
 &\leq \phi(u, u_n) - \phi(u, x_n) \\
 &\leq k_n \phi(u, u_n) - \phi(u, x_n) \\
 &\leq k_n^2 \phi(u, x_{n-1}) - \phi(u, x_n).
 \end{aligned}$$

Since  $\{\phi(u, x_n)\}$  converges, we have

$$\lim_{n \rightarrow \infty} g_1(\|x_n - u_n\|) = 0.$$

From the property of  $g_1$ , we have

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0.$$

Since  $J$  is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|Jx_n - Ju_n\| = 0.$$

From  $r_n \geq a$ , we have

$$\lim_{n \rightarrow \infty} \left\| \frac{Jx_n - Ju_n}{r_n} \right\| = 0.$$

By  $x_n = T_{r_n}u_n$ , we have

$$\tau(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \forall y \in C.$$

As in the proof of Theorem 1, we have  $\hat{x} \in GMEP(f, A, \varphi)$ . Therefore,  $\hat{x} \in F(T) \cap GMEP(f, A, \varphi)$ . Let  $y_n = \Pi_{F(T) \cap GMEP(f, A, \varphi)} x_n$ . From Lemma 2 and  $\hat{x} \in F(T) \cap GMEP(f, A, \varphi)$ , we have

$$\langle y_{n_k} - \hat{x}, Jx_{n_k} - Jy_{n_k} \rangle \geq 0.$$

From proposition 17, we also have that  $\{y_n\}$  converges strongly to  $z \in F(T) \cap GMEP(f, A, \varphi)$ , since  $J$  is weakly sequentially continuous, as  $k \rightarrow \infty$  we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle \geq 0. \quad (21)$$

On the other hand, since  $J$  is monotone, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle \leq 0. \quad (22)$$

Hence by (21) and (22), we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle = 0.$$

From the strict convexity of  $E$ , we have

$$z = \hat{x}.$$

Therefore,  $\{x_n\}$  converges weakly to  $\hat{x} \in F(T) \cap GMEP(f, A, \varphi)$ , and we have

$$\hat{x} = \lim_{n \rightarrow \infty} \Pi_{F(T) \cap GMEP(f, A, \varphi)} x_n.$$

□

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