Osward *et al. Journal of the Egyptian Mathematical Society* (2019) 27:43
https://doi.org/10.1186/s42787-019-0051-8

Journal of the Egyptian Mathematical Society

ORIGINAL RESEARCH **Open Access**

Approximation of common solutions for a fixed point problem of asymptotically nonexpansive mapping and a generalized equilibrium problem in Hilbert space

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Abstract
In this paper, we introduce an iterative algorithm to approximate a common solution of a generalized equilibrium problem and a fixed point problem for an asymptotically nonexpansive mapping in a real Hilbert space. We prove that the sequences generated by the iterative algorithm converge strongly to a common solution of the generalized equilibrium problem and the fixed point problem for an asymptotically nonexpansive mapping. The results presented in this paper extend and generalize many previously known results in this research area. Some applications of main results are also provided.

Keywords: Fixed points, Generalized equilibrium problem, Asymptotically nonexpansive mapping, Iterative algorithm

Mathematics Subject Classification 2010: 47J05, 47J20, 47H09.

Introduction

Throughout the paper unless otherwise stated, let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. Let C be a nonempty closed convex subsets of H. Let $\{x_n\}$ be a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H$. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

Such a P_C is called the metric projection from H onto C.

A mapping $T : C \to C$ is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ \forall x, y \in C.$$

T is said to be a uniformly k-Lipschitzian for a positive constant k if

$$\|T^n x - T^n y\| \leq k\|x - y\|, \forall x, y \in C, \forall n \in \mathbb{N}.$$

If $k_n = 1, \forall n \in \mathbb{N}$, then T is said to be a nonexpansive mapping. A point $x \in X$ is called a fixed point for T if $Tx = x$.

The fixed point problem (in short, FPP) for the mapping $T : C \to C$ is to find $x \in C$ such that

$$Tx = x. \tag{1}$$

The solution set of FPP (1) is denoted by $F(T)$, that is,

$$F(T) = \{x \in C : Tx = x\}.$$

Let $F : C \times C \to \mathbb{R}$ be a bifunction and $A : C \to H$ be a nonlinear mapping. The generalized equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{2}$$

The set of the solution of the problem (2) is denoted by $EP(F, A)$, that is,

$$EP(F, A) = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}.$$

If $A \equiv 0$ in (2), then problem (2) reduces to the equilibrium problem of finding an element $z \in C$ such that,

$$F(z, y) \geq 0, \forall y \in C. \tag{3}$$

The set of solutions of problem (3) is denoted by $EP(F)$.

If $F \equiv 0$ in (2), then the generalized equilibrium problem (2) is reduced to finding a point $z \in C$ such that,

$$\langle Az, y - z\rangle \geq 0, \forall y \in C, \tag{4}$$

which is called the classical variational inequality problem. The set of solution of the problem (4) is denoted by $VI(C, A)$.

If we define $F(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, then $z \in EP(F)$ if and only if $\langle Az, y - z\rangle \geq 0$ for all $y \in C$ and hence $z \in VI(C, A)$.

The problem (2) is very general in the sense that it includes many special cases such as optimization problems, variational inequalities, minimax problems, and the Nash equilibrium problem in noncooperative games; see Blum and Oettli [2], Kazmi and Rizvi [3], Meche et al.[4], Moudafi and Théra [5], Zegeye et al. [6], and the references therein.

Throughout this paper, let us assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$;

$$\lim_{t\downarrow 0} sup\, F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Definition 1 *A mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that,*

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

Remark 1 *Every α-inverse strong monotone mapping is $\frac{1}{\alpha}$-Lipschitz mapping; however, the converse may not hold.*

Takahashi and Takahashi [7] obtained the following strong convergence theorem to find a common solution of generalized equilibrium problem and the fixed point problem of a nonexpansive mapping in a Hilbert space.

Theorem 1 [7] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $A : C \to H$ be an α-inverse strongly monotone mapping, and let $T : C \to C$ be a nonepansive mapping such that $F(T) \cap EP(F, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequence generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T[\alpha_n u + (1 - \alpha_n)z_n], \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy

1. $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
2. $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0$, *and*
3. $0 < c \leq \beta_n \leq d < 1, 0 < a \leq \lambda_n \leq b < 2\alpha$.

Then, $\{x_n\}$ converges strongly to $z = P_{F(T)\cap EP(F,A)}(u)$.

In this paper, motivated by Takahashi and Takahashi [7], we construct an iterative algorithm for approximating a common solution of a generalized equilibrium problem and the fixed point problem for asymptotically nonexpansive mapping. It is also proved that the proposed algorithm converges strongly to a common solution.

Preliminaries

We now introduce preliminaries which will be used in this paper.

Recall that a mapping $f : C \to C$ is called a contraction mapping if there exists $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho\|x - y\|, \forall x, y \in C.$$

Lemma 1 [2] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Lemma 2 [8] *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying (A1), (A2), (A3), and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Furthermore, if

$$T_r x = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\},$$

then the following hold:

(1) T_r is single valued,

(2) T_r *is firmly non-expansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \forall x, y \in H,$$

(3) $F(T_r) = EP(F)$,

(4) $EP(F)$ *is closed and convex.*

Remark 2 *Replacing* x *with* $x - rAx \in H$ *in Lemma 1, there exists* $z \in C$*, such that*

$$F(z, y) + \langle Ax, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Definition 2 [9] *Let* C *be a closed convex subset of a Hilbert space* H*. A mapping* $T : C \to C$ *is called asymptotically regular at* x *if and only if,*

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0.$$

Lemma 3 [10] *Let* $F : C \to C$ *be a bifunction satisfying the conditions (A1) and (A2). Let* T_r *and* T_s *be defined as in Lemma 2 with* $r, s > 0$*. For any* $x, y \in H$*, then*

$$\|T_r y - T_s x\| \leq \|y - x\| + |\frac{r - s}{r}| \|T_r y - y\|.$$

Lemma 4 [11] *Let* $\{\delta_n\}$ *be a sequence of non negative real numbers, satisfying*

$$\delta_{n+1} \leq (1 - s_n)\delta_n + s_n\beta_n + \gamma_n, \forall n \geq 0,$$

where $\{s_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *satisfies the conditions:*

(i) $\{s_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} s_n = \infty$ *or equivalently,* $\prod_{n=1}^{\infty}(1 - s_n) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$,

(iii) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n \leq \infty$.

Then,

$$\lim_{n\to\infty} \delta_n = 0.$$

Lemma 5 [12] *Let* T *be an asymptotically nonexpansive mapping on a closed and convex subset* C *of a real Hilbert space* H*. Then,* $I - T$ *is demiclosed at* 0*. That is, for a sequence* $\{x_n\}$ *in* C*, if* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$*, then* $x \in F(T)$*.*

Lemma 6 [13] *Let* H *be a real Hilbert space. Then, for any given* $x, y \in H$*, we have the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 7 [14] *Let* $\{t_n\}$ *be a sequence of nonnegative real numbers such that*

$$t_{n+1} \leq (1 - a_n)t_n + a_n\beta_n, \quad n \geq 0$$

where $\{a_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\beta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(C1) $\sum_{n=0}^{\infty} a_n = \infty$ *or equivalently* $\prod_{n=0}^{\infty}(1 - a_n) = 0$,

(C2) $\limsup_{n\to\infty} \beta_n \leq 0$.

Then

$$\lim_{n\to\infty} t_n = 0.$$

Main results

Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $A : C \to H$ be α-inverse strongly monotone mapping. Then, it follows from Lemma 2 that for each $r > 0$ and $x \in H$ there is $w \in C$ such that

$$T_r(x) = \{w\}$$

where $T_r x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} = \{w\}$, so that we identify $T_r x$ as simply w.

Let $f : C \to C$ be ρ-contraction mapping and let $T : C \to C$ be asymptotically nonexpansive mapping. Let $\{\alpha_n\} \subset [0, 1]$ and $\lambda_n \in (0, 2\alpha)$. For any $x_1 \in C$, we find $z_1 \in C$ such that

$$z_1 = T_{\lambda_1}(x_1 - \lambda_1 A x_1).$$

Then, we can compute $x_2 \in C$ by

$$x_2 = \alpha_1 f(x_1) + (1 - \alpha_1) T z_1.$$

Also, we can find $z_2 \in C$ such that

$$z_2 = T_{\lambda_2}(x_2 - \lambda_2 A x_2).$$

After that, we can compute $x_3 \in C$ by

$$x_3 = \alpha_2 f(x_2) + (1 - \alpha_2) T^2 z_2.$$

Inductively, we can generate the sequence $\{x_n\} \subset C$ as follows:

$$\begin{cases} x_1 \in C, \\ z_n = T_{\lambda_n}(x_n - \lambda_n A x_n), n = 1, 2, 3, ... \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T^n z_n, n = 1, 2, 3, ... \end{cases} \tag{5}$$

Now, we state and prove our convergence theorem as follows:

Theorem 2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $f : C \to C$ be ρ-contraction mapping, $A : C \to H$ be an α-inverse strongly monotone mapping, and $T : C \to C$ be asymptotically nonexpansive mapping. Assume that T is asymptotically regular on C such that $F(T) \cap EP(F, A) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(ii) $0 < a \leq \lambda_n \leq b < 2\alpha,$

(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0,$

(iv) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0.$

For $x_1 \in C$, if $\{x_n\}$ is the sequence defined by the iterative scheme (5), then $\{x_n\}$ converges strongly to $z = P_{F(T)\cap EP(F,A)} f(z)$.

Proof We first show that $\{x_n\}$ is bounded. Let $z \in F(T) \cap EP(F, A)$. Since $z = T_{\lambda_n}(z - \lambda_n Az)$, A is α-inverse strongly monotone and $0 < \lambda_n \leq 2\alpha$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|z_n - z\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\
&\leq \|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 \\
&= \|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2 \\
&\leq \|x_n - z\|^2.
\end{aligned}$$

Hence, we have

$$\|z_n - z\| \leq \|x_n - z\|. \tag{6}$$

Take $\epsilon \in (0, 1 - \rho)$. Since $\frac{k_n - 1}{\alpha_n} \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that

$$(k_n - 1) < \epsilon\alpha_n \text{ for all } n \geq \mathbb{N}.$$

From (5) and (6) it follows that, for all $n > N$

$$\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n f(x_n) + (1 - \alpha_n)T^n z_n - z\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(z) + \alpha_n f(z) - \alpha_n z + \alpha_n z + (1 - \alpha_n)T^n z_n - z\| \\
&= \|\alpha_n(f(x_n) - f(z)) + \alpha_n(f(z) - z) + (1 - \alpha_n)(T^n z_n - z)\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\|T^n z_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)\|T^n z_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)k_n\|z_n - z\| \\
&\leq \alpha_n\rho\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)k_n\|x_n - z\| \\
&= (1 - \alpha_n(1 - \rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + (1 - \alpha_n)(k_n - 1)\|x_n - z\| \\
&\leq (1 - \alpha_n(1 - \rho))\|x_n - z\| + \alpha_n\|f(z) - z\| + \alpha_n\epsilon\|x_n - z\| \\
&= (1 - \alpha_n(1 - \rho - \epsilon))\|x_n - z\| + \alpha_n\|f(z) - z\| \\
&\leq \max\left\{\|x_n - z\|, \frac{1}{1 - \rho - \epsilon}\|f(z) - z\|\right\}.
\end{aligned}$$

By induction, we see that, for all $n \geq 1$

$$\|x_n - z\| \leq \max\{\|x_1 - z\|, \frac{1}{1 - \rho - \epsilon}\|f(z) - z\|\}.$$

So $\{x_n\}$ is bounded, hence $\{Ax_n\}$, $\{f(x_n)\}$, $\{z_n\}$ and $\{T^n z_n\}$ are bounded.
Next, we have to prove that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Since $I - \lambda_n A$ is non-expansive and by Lemma 3, then we have

$$\begin{aligned}
\|z_{n+1} - z_n\| = &\ \|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - T_{\lambda_n}(x_n - \lambda_n Ax_n)\| \\
\leq &\ \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_n Ax_n)\| \\
&+ \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\
= &\ \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n) + (\lambda_n - \lambda_{n+1})Ax_n\| \\
&+ \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\
\leq &\ \|(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_n - \lambda_{n+1}Ax_n)\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \\
&+ \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\
\leq &\ \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| \\
&+ \left|\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}}\right| \|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\| \\
\leq &\ \|x_{n+1} - x_n\| + |\lambda_n - \lambda_{n+1}|\|Ax_n\| + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}} P_{n+1} \qquad (7)
\end{aligned}$$

where by $P_{n+1} = \sup\{\|T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1}) - (x_{n+1} - \lambda_{n+1}Ax_{n+1})\|\}$.

On the other hand, from $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n)$ and $z_{n+1} = T_{\lambda_{n+1}}(x_{n+1} - \lambda_{n+1}Ax_{n+1})$, we have

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C. \qquad (8)$$

and

$$F(z_{n+1}, y) + \langle Ax_{n+1}, y - z_{n+1}\rangle + \frac{1}{\lambda_{n+1}}\langle y - z_{n+1}, z_{n+1} - x_{n+1}\rangle \geq 0, \forall y \in C. \qquad (9)$$

Putting $y = z_{n+1}$ in (8) and $y = z_n$ in (9), we have

$$F(z_n, z_{n+1}) + \langle Ax_n, z_{n+1} - z_n\rangle + \frac{1}{\lambda_n}\langle z_{n+1} - z_n, z_n - x_n\rangle \geq 0. \qquad (10)$$

and

$$F(z_{n+1}, z_n) + \langle Ax_{n+1}, z_n - z_{n+1}\rangle + \frac{1}{\lambda_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - x_{n+1}\rangle \geq 0. \qquad (11)$$

So, from $(A2)$, we have,

$$\langle Ax_{n+1} - Ax_n, z_n - z_{n+1}\rangle + \left\langle z_{n+1} - z_n, \frac{z_n - x_n}{\lambda_n} - \frac{z_{n+1} - x_{n+1}}{\lambda_{n+1}}\right\rangle \geq 0.$$

And hence,

$$\begin{aligned}
0 \leq &\ \left\langle z_n - z_{n+1}, \lambda_n(Ax_{n+1} - Ax_n) + \frac{\lambda_n}{\lambda_{n+1}}(z_{n+1} - x_{n+1}) - (z_n - x_n)\right\rangle \\
= &\left\langle z_{n+1} - z_n, z_n - z_{n+1} + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right) z_{n+1} + (x_{n+1} - \lambda_n Ax_{n+1})\right\rangle \\
&+ \left\langle z_{n+1} - z_n, (\lambda_n Ax_n - x_n) - x_{n+1} + \frac{\lambda_n}{\lambda_{n+1}} x_{n+1}\right\rangle \\
= &\left\langle z_{n+1} - z_n, z_n - z_{n+1} + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(z_{n+1} - x_{n+1})\right\rangle \\
&+ \langle z_{n+1} - z_n, (x_{n+1} - \lambda_n Ax_{n+1}) - (x_n - \lambda_n Ax_n)\rangle.
\end{aligned}$$

It then follows that

$$\|z_{n+1} - z_n\|^2 \le \|z_{n+1} - z_n\| \left\{ \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \right\}$$

And so, we have

$$\|z_{n+1} - z_n\| \le \left| 1 - \frac{\lambda_n}{\lambda_{n+1}} \right| \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|. \tag{12}$$

Using condition (ii), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \|x_{n+1} - x_n\| + \frac{1}{\lambda_{n+1}} |\lambda_{n+1} - \lambda_n| \|z_{n+1} - x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + \frac{1}{a} |\lambda_{n+1} - \lambda_n| M, \end{aligned} \tag{13}$$

where $M = \sup_{n \ge 1} \|z_n - x_n\|$. Hence, we have

$$\|z_n - z_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{a} |\lambda_n - \lambda_{n-1}| M. \tag{14}$$

Consider

$$\begin{aligned} \|T^n z_n - T^{n-1} z_{n-1}\| &\le \|T^n z_n - T^n z_{n-1}\| + \|T^n z_{n-1} - T^{n-1} z_{n-1}\| \\ &\le k_n \|z_n - z_{n-1}\| + \|T^n z_{n-1} - T^{n-1} z_{n-1}\|. \end{aligned} \tag{15}$$

From (5), (14) and (15), we have that

$$\begin{aligned} \|x_{n+1} - x_n\| = &\ \|\alpha_n f(x_n) + (1-\alpha_n) T^n z_n - \alpha_{n-1} f(x_{n-1}) - (1-\alpha_{n-1}) T^{n-1} z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \big(\|f(x_{n-1}\| + \|T^{n-1} z_{n-1}\| \big) \\ &+ (1-\alpha_n) \|T^n z_n - T^{n-1} z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + (1-\alpha_n) \|T^n z_n - T^{n-1} z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + (1-\alpha_n) k_n \|z_n - z_{n-1}\| \\ &+ (1-\alpha_n) \|T^n z_{n-1} - T^{n-1} z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + (1-\alpha_n)(k_n - 1) \|z_n - z_{n-1}\| \\ &+ (1-\alpha_n) \|T^n z_{n-1} - T^{n-1} z_{n-1}\| + (1-\alpha_n) \|z_n - z_{n-1}\| \\ &\le \alpha_n \rho \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + (k_n - 1) \|x_n - x_{n-1}\| \\ &+ (k_n - 1) \frac{1}{a} |\lambda_n - \lambda_{n-1}| M + \|T^n z_{n-1} - T^{n-1} z_{n-1}\| \\ &+ (1-\alpha_n) \left[\|x_n - x_{n-1}\| + \frac{1}{a} |\lambda_n - \lambda_{n-1}| M \right] \\ &\le (1-\alpha_n(1-\rho-\epsilon)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + \frac{\epsilon \alpha_n}{a} |\lambda_n - \lambda_{n-1}| M \\ &+ \|T^n z_{n-1} - T^{n-1} z_{n-1}\| + \frac{(1-\alpha_n)}{a} |\lambda_n - \lambda_{n-1}| M, \\ &\le (1-\alpha_n(1-\rho-\epsilon)) \|x_n - x_{n-1}\| + \alpha_n (1-\rho-\epsilon) \frac{|\lambda_n - \lambda_{n-1}|}{a} M \\ &+ (1+\alpha_n(2\epsilon+\rho)) \frac{|\lambda_n - \lambda_{n-1}|}{a} M + |\alpha_n - \alpha_{n-1}| K \\ &+ \|T^n z_{n-1} - T^{n-1} z_{n-1}\|, \end{aligned}$$

where $K = \sup\{\|f(x_n)\| + \|T^n z_n\|\}$. Put $s_n = \alpha_n(1-\rho-\epsilon)$, $\beta_n = \frac{|\lambda_n - \lambda_{n-1}|}{a} M$ and $\gamma_n = (1+\alpha_n(2\epsilon+\rho)) \frac{|\lambda_n - \lambda_{n-1}|}{a} M + |\alpha_n - \alpha_{n-1}| K + \|T^n z_{n-1} - T^{n-1} z_{n-1}\|$. Then,

$$\|x_{n+1} - x_n\| \le (1-s_n) \|x_n - x_{n-1}\| + s_n \beta_n + \gamma_n$$

Using Lemma 4, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{16}$$

Further by (13) with the condition that $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$, we get

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0. \tag{17}$$

Since $x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})T^{n-1} z_{n-1}$, we have

$$\begin{aligned}
\|x_n - T^n z_n\| &\le \|x_n - T^{n-1}z_{n-1}\| + \|T^{n-1}z_{n-1} - T^n z_n\| \\
&\le \|x_n - T^{n-1}z_{n-1}\| + \|T^{n-1}z_{n-1} - T^n z_{n-1}\| + \|T^n z_{n-1} - T^n z_n\| \\
&\le \alpha_{n-1}\|f(x_{n-1}) - T^{n-1}z_{n-1}\| \\
&\quad + \|T^{n-1}z_{n-1} - T^n z_{n-1}\| + k_n\|z_{n-1} - z_n\|.
\end{aligned}$$

From (17) with $\alpha_n \to 0$ as $n \to \infty$ and T is asymptotically regular on C. It follows that

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{18}$$

Now, we have to prove that

$$\lim_{n\to\infty} \|T x_n - x_n\| = 0.$$

To show this, we first prove that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

With the fact that A is α-inverse strongly monotone, let us consider the following:

$$\begin{aligned}
\|z_n - z\|^2 &= \|T_{\lambda_n}(I - \lambda_n A)x_n - T_{\lambda_n}(I - \lambda_n A)z\|^2 \\
&\le \|(I - \lambda_n A)x_n - (I - \lambda_n A)z\|^2 \\
&= \|(x_n - z) - \lambda_n(Ax_n - Az\|^2 \\
&= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2 \\
&\le \|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2 \\
&= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2.
\end{aligned} \tag{19}$$

From the convexity of $\|.\|^2$, (18), and (19), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) - \alpha_n z + \alpha_n z + (1-\alpha_n)T^n z_n - z\|^2 \\
&= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(T^n z_n - z)\|^2 \\
&\le \alpha_n\|(f(x_n) - z\|^2 + (1-\alpha_n)\|T^n z_n - z\|^2 \\
&\le \alpha_n\|(f(x_n) - z\|^2 + (1-\alpha_n)k_n^2\|z_n - z\|^2 \qquad (20)\\
&\le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)k_n^2[\,\|x_n - z\|^2 \\
&\quad + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2] \\
&= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)k_n^2\|x_n - z\|^2 \\
&\quad + \lambda_n(\lambda_n - 2\alpha)(1-\alpha_n)k_n^2\|Ax_n - Az\|^2 \qquad (21)\\
&= \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\left(k_n^2 - 1\right)\|x_n - z\|^2 \\
&\quad + (1-\alpha_n)\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)(1-\alpha_n)k_n^2\|Ax_n - Az\|^2 \\
&\le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\left(k_n^2 - 1\right)\|x_n - z\|^2 + \|x_n - z\|^2 \\
&\quad + \lambda_n(\lambda_n - 2\alpha)(1-\alpha_n)k_n^2\|Ax_n - Az\|^2 \qquad (22)
\end{aligned}$$

which implies that

$$\lambda_n(2\alpha - \lambda_n)(1-\alpha_n)k_n^2\|Ax_n - Az\|^2 \le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(k_n^2 - 1)\|x_n - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} k_n = 1$ and both $\{f(x_n)\}$ and $\{x_n\}$ are bounded by (16), we have

$$\lim_{n\to\infty} \|Ax_n - Az\| = 0. \tag{23}$$

Since $(I - \lambda_n A)$ is non-expansive and by Lemma 2, we have

$$\begin{aligned}
\|z_n - z\|^2 =& \|T_{\lambda_n}(x_n - \lambda_n Ax_n) - T_{\lambda_n}(z - \lambda_n Az)\|^2 \\
\le& \langle z_n - z, (x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\rangle \\
=& \frac{1}{2}\left(\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az)\|^2 + \|z_n - z\|^2\right) \\
&- \frac{1}{2}\left(\|(x_n - \lambda_n Ax_n) - (z - \lambda_n Az) - (z_n - z)\|^2\right) \\
=& \frac{1}{2}\left(\|(I - \lambda_n A)x_n - (I - \lambda_n A)z\|^2 + \|z_n - z\|^2\right) \\
&- \frac{1}{2}\left(\|(x_n - z_n) - \lambda_n(Ax_n - Az)\|^2\right) \\
\le& \frac{1}{2}(\|x_n - z\|^2 + \|z_n - z\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Az)\|^2) \\
\le& \frac{1}{2}\left(\|x_n - z\|^2 + \|z_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Az\rangle\right) \\
&- \frac{1}{2}\lambda_n^2\|Ax_n - Az\|^2
\end{aligned}$$

which implies that

$$\|z_n - z\|^2 \le \|x_n - z\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\langle x_n - z_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2. \tag{24}$$

From (20) and (24), we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 \le& \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)k_n^2\left(\|x_n - z\|^2 - \|x_n - z_n\|^2\right) \\
&+ 2\lambda_n(1-\alpha)k_n^2\left(\langle x_n - z_n, Ax_n - Az\rangle - \lambda_n^2\|Ax_n - Az\|^2\right) \\
\le& \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)k_n^2\|x_n - z\|^2 - (1-\alpha_n)k_n^2\|x_n - z_n\|^2 \\
&+ 2(1-\alpha_n)k_n^2\lambda_n\|x_n - z_n\|\|Ax_n - Az\| - (1-\alpha_n)k_n^2\lambda_n^2\|Ax_n - Az\|^2 \\
\le& \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)k_n^2\|x_n - z\|^2 - (1-\alpha_n)k_n^2\|x_n - z_n\|^2 \\
&+ 2(1-\alpha_n)k_n^2\lambda_n\|x_n - z_n\|\|Ax_n - Az\| \\
=& \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(k_n^2 - 1)\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\
&- (1-\alpha_n)k_n^2\|x_n - z_n\|^2 + 2(1-\alpha_n)k_n^2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|) \\
\le& \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)(k_n^2 - 1)\|x_n - z\|^2 + \|x_n - z\|^2 \\
&- (1-\alpha_n)k_n^2\|x_n - z_n\|^2 + 2(1-\alpha_n)k_n^2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.
\end{aligned}$$

Hence,

$$(1-\alpha_n)k_n^2\|x_n - z_n\|^2 \le \alpha_n\|f(x_n) - z\|^2 + (1-\alpha_n)\left(k_n^2 - 1\right)\|x_n - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2(1-\alpha_n)k_n^2\lambda_n\|x_n - z_n\|\|Ax_n - Az\|.$$

Since $\alpha_n \to 0$, $k_n \to 1$ as $n \to \infty$ and $\{x_n\}$, $\{z_n\}$ are bounded with $\lim_{n\to\infty} \|x_n - z\|^2 - \|x_{n+1} - z\|^2 = 0$, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{25}$$

Combining (16) and (25) we have, $\|z_n - x_{n+1}\| \le \|z_n - x_n\| + \|x_n - x_{n+1}\|$ which implies that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0. \tag{26}$$

Since $\|T^n z_n - z_n\| \le \|T^n z_n - x_n\| + \|x_n - z_n\|$ and from (18) and (25)

$$\lim_{n\to\infty} \|T^n z_n - z_n\| = 0. \tag{27}$$

Let $k = \sup_{n\in\mathbb{N}} k_n < \infty$. Consequently, by (17) and (27)

$$\begin{aligned}
\|Tz_n - z_n\| \le\ & \|Tz_n - T^{n+1}z_n\| + \|T^{n+1}z_n - T^{n+1}z_{n+1}\| \\
& + \|T^{n+1}z_{n+1} - z_{n+1}\| + \|z_{n+1} - z_n\| \\
\le\ & k_1\|z_n - T^n z_n\| + k_{n+1}\|z_n - z_{n+1}\| \\
& + \|T^{n+1}z_{n+1} - z_{n+1}\| + \|z_{n+1} - z_n\| \\
=\ & k_1\|z_n - T^n z_n\| + (k_{n+1} + 1)\|z_n - z_{n+1}\| + \|T^{n+1}z_{n+1} - z_{n+1}\| \\
\le\ & k\|z_n - T^n z_n\| + (k + 1)\|z_n - z_{n+1}\| + \|T^{n+1}z_{n+1} - z_{n+1}\|
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|Tz_n - z_n\| = 0.$$

Further, we have the following result:

$$\begin{aligned}
\|Tx_n - x_n\| \le\ & \|Tx_n - Tz_n\| + \|Tz_n - z_n\| + \|z_n - x_n\| \\
\le\ & k_1\|x_n - z_n\| + \|Tz_n - z_n\| + \|z_n - x_n\| \\
\le\ & (k_1 + 1)\|x_n - z_n\| + \|Tz_n - z_n\|
\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{28}$$

Since $P_{F(T)\cap EP(F,A)}f : C \to C$ is a ρ-contraction mapping, therefore, by Banach contraction principle, there exists a unique $z_0 \in F(T) \cap EP(F,A)$ such that $z_0 = P_{F(T)\cap EP(F,A)}f(z_0)$. We shall show that

$$\lim_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle \le 0. \tag{29}$$

Since $\{x_n\}$ is bounded sequence, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z_0) - z_0, x_n - z_0\rangle = \lim_{i\to\infty} \langle f(z_0) - z_0, x_{n_i} - z_0\rangle. \tag{30}$$

Without loss of generality, we may assume that $x_{n_i} \rightharpoonup \omega$. Since C is closed and convex, C is weakly closed. So, we have $\omega \in C$. Now, we will show that $\omega \in F(T)$. In fact, since $x_{n_i} \rightharpoonup \omega$ and $x_n - Tx_n \to 0$ by Lemma 5, we find that $\omega \in F(T)$.

Next, we show that $\omega \in EP(F,A)$. From (25), we have $z_{n_i} \rightharpoonup \omega$.

Since $z_n = T_{\lambda_n}(x_n - \lambda_n Ax_n)$, that is $F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0, \forall y \in C$. From $(A2)$, we have

$\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq F(y, z_n), \forall y \in C.$
Replacing n with n_i in the above inequality, we have,

$$\langle Ax_{n_i}, y - z_{n_i}\rangle + \frac{1}{\lambda_{n_i}}\langle y - z_{n_i}, z_{n_i} - x_{n_i}\rangle \geq F(y, z_{n_i}). \tag{31}$$

Put $z_t = ty + (1-t)\omega$ for all $t \in (0, 1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (31) we have

$$\begin{aligned}\langle z_t - z_{n_i}, Az_t\rangle \geq\ & \langle z_t - z_{n_i}, Az_t\rangle - \langle z_t - z_{n_i}, Ax_{n_i}\rangle - \langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle + F(z_t, z_{n_i}) \\ =\ & \langle z_t - z_{n_i}, Az_t - Az_{n_i}\rangle + \langle z_t - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle \\ -\ & \langle z_t - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle + F(z_t, z_{n_i}).\end{aligned}$$

Since $\lim_{n\to\infty}\|z_{n_i} - x_{n_i}\| = 0$, we have $\lim_{n\to\infty}\|Az_{n_i} - Ax_{n_i}\| = 0$.

Further from monotonicity of A, we have $\langle z_t - z_{n_i}, Az_t - Az_{n_i}\rangle \geq 0$. It follows from $(A4)$ that

$$F(z_t, \omega) \leq \lim_{n\to\infty} F(z_t, z_{n_i}) \leq \lim_{n\to\infty}\langle Az_t, z_t - z_{n_i}\rangle = \langle Az_t, z_t - \omega\rangle \tag{32}$$

From $(A1)$, $(A4)$, and (32), we have

$$\begin{aligned}0 = F(z_t, z_t) \leq\ & tF(z_t, y) + (1-t)F(z_t, \omega) \\ \leq\ & tF(z_t, y) + (1-t)\langle z_t - \omega, Az_t\rangle.\end{aligned}$$

But $z_t - \omega = ty + (1-t)\omega - \omega = t(y - \omega)$. So, we have the following $0 = F(z_t, z_t) \leq tF(z_t, y) + (1-t)t\langle y - \omega, Az_t\rangle$ and hence $0 \leq F(z_t, y) + (1-t)\langle y - \omega, Az_t\rangle$. Letting $t \to 0$, we have for each $y \in C$,

$$\begin{aligned}& 0 \leq\ F(\omega, y) + \langle y - \omega, A\omega\rangle. \\ & \Longrightarrow\ \omega \in EP(F, A).\end{aligned} \tag{33}$$

Since $\omega \in F(T) \cap EP(F, A)$, from (30) and the property of metric projection, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle =\ & \lim_{i\to\infty}\langle f(z_0) - z_0, x_{n_i} - z_0\rangle \\ =\ & \langle f(z_0) - z_0, \omega - z_0\rangle \leq 0.\end{aligned} \tag{34}$$

Finally, we prove that $\lim_{n\to\infty}\|x_n - z_0\| = 0$.

From (5) and Lemma 6, we obtain

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &= \|\alpha_n(f(x_n)-z_0)+(1-\alpha_n)(T^n z_n-z_0)\|^2\\
&\leq (1-\alpha_n)^2\|T^n z_n-z_0\|^2+2\langle\alpha_n(f(x_n)-z_0),x_{n+1}-z_0\rangle\\
&= (1-\alpha_n)^2\|T^n z_n-z_0\|^2+2\alpha_n\langle f(x_n)-z_0,x_{n+1}-z_0\rangle\\
&\leq [(1-\alpha_n)k_n]^2\|z_n-z_0\|^2+2\alpha_n\langle f(x_n)-z_0,x_{n+1}-z_0\rangle\\
&= [(1-\alpha_n)k_n]^2\|z_n-z_0\|^2+2\alpha_n\langle f(x_n)-f(z_0),x_{n+1}-z_0\rangle\\
&\quad+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&\leq [(1-\alpha_n)k_n]^2\|x_n-z_0\|^2+2\alpha_n\langle f(x_n)-f(z_0),x_{n+1}-z_0\rangle\\
&\quad+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&\leq [(1-\alpha_n)k_n]^2\|x_n-z_0\|^2+2\alpha_n\rho\|x_n-z_0\|\|x_{n+1}-z_0\|\\
&\quad+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&\leq [(1-\alpha_n)k_n]^2\|x_n-z_0\|^2+\alpha_n\rho(\|x_n-z_0\|^2+\|x_{n+1}-z_0\|^2)\\
&\quad+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&= [(1-\alpha_n)(k_n-1)+(1-\alpha_n)]^2\|x_n-z_0\|^2+\alpha_n\rho\|x_n-z_0\|^2\\
&\quad+\alpha_n\rho\|x_{n+1}-z_0\|^2+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&= [1-(2-\rho)\alpha_n+\alpha_n^2+(1-\alpha_n)^2(1-k_n)^2+\alpha_n\rho\|x_{n+1}-z_0\|^2\\
&\quad+2(1-\alpha_n)^2(k_n-1)]\|x_n-z_0\|^2+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&\leq [1-(2-\rho)\alpha_n+\alpha_n^2+(1-k_n)^2+2(k_n-1)]\|x_n-z_0\|^2\\
&\quad+\alpha_n\rho\|x_{n+1}-z_0\|^2+2\alpha_n\langle f(z_0)-z_0,x_{n+1}-z_0\rangle.
\end{aligned}$$

Let $P_n=\sup_{n\in\mathbb{N}}\|x_n-z_0\|^2$, so now we have

$$\begin{aligned}
\|x_{n+1}-z_0\|^2 &\leq \frac{1-(2-\rho)\alpha_n}{1-\rho\alpha_n}\|x_n-z_0\|^2+\frac{\alpha_n^2+(k_n-1)^2+2(k_n-1)}{1-\rho\alpha_n}P_n\\
&\quad+\frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&= \left[1-\frac{(2-\rho)\alpha_n}{1-\rho\alpha_n}\right]\|x_n-z_0\|^2+\frac{\alpha_n^2+(k_n-1)^2+2(k_n-1)}{1-\rho\alpha_n}P_n\\
&\quad+\frac{2\alpha_n}{1-\rho\alpha_n}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle\\
&= (1-a_n)\|x_n-z_0\|^2+a_n\beta_n,
\end{aligned}$$

where $\beta_n=\frac{\alpha_n^2+(k_n-1)^2+2(k_n-1)}{2(1-\rho)\alpha_n}P_n+\frac{1}{1-\rho}\langle f(z_0)-z_0,x_{n+1}-z_0\rangle$ and $a_n=\frac{2(1-\rho)\alpha_n}{1-\rho\alpha_n}$. Since $\lim_{n\to\infty}a_n=0$, $\sum_{n=0}^{\infty}a_n=\infty$, and $\limsup_{n\to\infty}\beta_n\leq 0$ by (34), then by Lemma 7, we conclude that $\lim_{n\to\infty}\|x_n-z_0\|=0$. □

Applications

Using our main theorem (Theorem 2), we obtain strong convergence theorems in Hilbert space.

Theorem 3 *Let C be a nonempty closed convex and bounded subset of H. Let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $f: C\to C$ be ρ-contraction mapping, and let $T: C\to C$ be asymptotically nonexpansive mapping. Assume that T*

is asymptotically regular on C such that $F(T) \cap EP(F) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$
(ii) $0 < a \leq \lambda_n \leq b < 2\alpha,$
(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0,$
(iv) $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0.$

For $x_1 \in C$, if $\{x_n\}$ is the sequence defined by the iterative scheme (5), then $\{x_n\}$ converges strongly to $z = P_{F(T)\cap EP(F)} f(z)$.

Proof In theorem (2), put $A \equiv 0$. We obtain that $F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C$. Then, for all $\alpha \in (0, \infty)$, we have $\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2 = 0, \forall x, y \in C$. Thus, we obtain the desired result by Theorem 2. □

Theorem 4 *Let C be a nonempty closed convex and bounded subset of a real Hilbert space H. Let $f : C \to C$ be ρ-contraction mapping, A be an α-inverse strongly monotone mapping of C into H, and $T : C \to C$ be asymptotically nonexpansive mapping. Assume that T is asymptotically regular on C such that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy*

(i) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$
(ii) $0 < a \leq \lambda_n \leq b < 2\alpha,$
(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0,$
(iv) $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = 0.$

For $x_1 \in C$, if $\{x_n\}$ is the sequence defined by the iterative scheme (5), then $\{x_n\}$ converges strongly to $z = P_{F(T)\cap VI(C,A)} f(z)$.

Proof In Theorem 2, put $F \equiv 0$. Then, we obtain that $\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \forall y \in C, \forall n \in \mathbb{N}$.
This implies that $\langle x_n - \lambda_n Ax_n - z_n, z_n - y\rangle \geq 0, \forall y \in C$. So, we find that $z_n = P_C(x_n - \lambda_n Ax_n)$. Then, we obtain the desired result from Theorem 2. □

Browder and Patryshyn [9] introduced k- strictly pseudocontractive mapping which is as follows:
A mapping $S : C \to C$ is called k- strictly pseudocontractive if there exists $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \forall x, y \in C.$$

Putting $A = I - S$, we know that

$$\langle x - y, Ax - Ay\rangle \geq \frac{1-k}{2}\|Ax - Ay\|^2, \forall x, y \in C.$$

By using the above definition and Theorem 2, we can obtain the following theorem.

Theorem 5 *Let C be a nonempty closed convex and bounded subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying $(A1 - A4)$. Let $f : C \to C$ be*

ρ-contraction mapping, S be a k-strictly pseudo contractive mapping of C into itself, and $T : C \to C$ be asymptotically non-expansive mapping. Assume that T is asymptotically regular on C such that $F(T) \cap EP(F, A) \neq \emptyset$, where $A = I - S$. Let $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset [0, 1-k]$ satisfy

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < a \leq \lambda_n \leq b < 1 - k$,
(iii) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0$,
(iv) $\lim_{n\to\infty} \frac{k_n - 1}{\alpha_n} = 0$.

For $x_1 \in C$, if $\{x_n\}$ is the sequence defined by the iterative scheme (5), then $\{x_n\}$ converges strongly to $z = P_{F(T)\cap EP(F,A)} f(z)$.

Proof Since $A = I - S$ is $\frac{1-k}{2}$-inverse strongly monotone mapping. So, by Theorem 2, we obtain the desired result. □

Remark 3 *By replacing asymptotically nonexpansive mapping to nonexpansive single valued mapping, it gives an improved version of the main result due to Takahashi and Takahashi* [7].

Acknowledgments
The authors would like to thank and acknowledge Sida-Mathematics Project at the Univeversity of Dar es Salaam for their financial support through out the preparation of this article.

Authors' contributions
All authors equally contributed in the manuscript. All authors read and approved the final manuscript.

Funding
This project is funded by the Sida-Mathematics project at the University of Dar es Salaam, Tanzania.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

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Received: 16 May 2019 Accepted: 23 October 2019
Published online: 07 November 2019

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