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Coupled coincidence point technique and its application for solving nonlinear integral equations in RPOCbML spaces

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Abstract

The aim of this paper is to obtain some new important consequences related to coupled coincidence points via C-class functions in the context of a regular partial ordered complete b -metric-like space (for short, RPOCbML space); this space arises from combining the results of b -metric-like space with partial metric space and adding the regularity condition. Finally, we support our theoretical results by some examples and an application about finding an analytical solution for nonlinear integral equations.

Keywords: Coupled coincidence point, Metric-like spaces, Regular property, Nonlinear integral equations

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Introduction and elementary discussions

Fixed point theory is one of the most important branches of non-linear analysis because it contributes to many disciplines such as economics (maximization and minimization problems), engineering and game theory, mathematics, and others. One of the most important advances in mathematics, for example, is to find solutions for linear and non-linear systems and others. Due to the ease of this method, it became the most prevalent in functional analysis. This style emerged in 1922 when Banach [1] introduced his well-known principle and called it Banach contraction principle in metric space. In 1992, Matthews [2] circulated the principle of Banach to partial metric space as follows:

Definition 1 [2] *A mapping $p : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^+$, where Υ is a nonempty set, is said to be a partial metric on Υ if for any $a, c, e \in \Upsilon$ the following hypotheses hold:*

(P₁) $a = c$ if and only if $p(a, a) = p(c, c) = p(a, c)$;

(P₂) $p(a, a) \leq p(a, c)$;

(P₃) $p(a, c) = p(c, a)$;

(P₄) $p(a, e) \leq p(a, c) + p(c, e) - p(c, c)$.

Then the pair (Υ, p) is called a partial metric space.

The concept of b -metric space was illustrated by Bakhtin [3] in 1993 and Czerwik [4] in 1998. Since then, many papers have been published on the fixed point theory of various

classes of single-valued and multi-valued operators in b -metric spaces (see, [5–8] and references therein).

Definition 2 [4] *Let $\Upsilon \neq \emptyset$ be a set and $s \geq 1$ be a given real number. A function $\omega_b : \Upsilon \times \Upsilon \rightarrow [0, \infty)$ is a b -metric if and only if, for all $a, c, e \in \Upsilon$, the following conditions are satisfied:*

$$(b_1) \omega_b(a, c) = 0 \Leftrightarrow a = c;$$

$$(b_2) \omega_b(a, c) = \omega_b(c, a);$$

$$(b_3) \omega_b(a, c) \leq s[\omega_b(a, e) + \omega_b(e, c)].$$

The pair (Υ, ω_b) is called a b -metric space (with a coefficient $s \geq 1$).

Amini-Harandi [9] introduced the notion of metric-like spaces which is an interesting generalization of the spaces defined in [2] as follows:

Definition 3 [9] *Let Υ be a nonempty set, a mapping $\omega : \Upsilon \times \Upsilon \rightarrow \mathbb{R}^+$ is said to be a metric-like mapping on Υ if for all $a, c, e \in \Upsilon$ the following three conditions hold:*

$$(\omega_1) \omega(a, c) = 0 \Rightarrow a = c;$$

$$(\omega_2) \omega(a, c) = \omega(c, a);$$

$$(\omega_3) \omega(a, e) \leq \omega(a, c) + \omega(c, e).$$

Then, the pair (Υ, ω) is called a metric-like space.

A metric-like mapping on Υ satisfies all of the conditions of a metric except that $\omega(a, a)$ may be positive for $a \in \Upsilon$. Every partial metric space is a metric-like space but not conversely in general. For more details, see [9].

By considering the results of b -metric and metric-like spaces, the notion of b -metric-like space was presented by Alghamdi et al. [10]. They discussed some related fixed point consequences concerning with this space. Recently, Hussain et al. [11] examined topological structure of this space and presented some fixed point results in b -metric-like space. A lot of results on fixed points of mappings via certain contractive conditions in the mentioned spaces have been done (for example, see [11–13]).

Definition 4 [10] *Let $\Upsilon \neq \emptyset$ be a set and $s \geq 1$ be a given real number. A function $\omega_b : \Upsilon \times \Upsilon \rightarrow [0, \infty)$ is a b -metric-like space if and only if, for all $a, c, e \in \Upsilon$, the following conditions are realized:*

$$(b_1) \omega_b(a, c) = 0 \Rightarrow a = c;$$

$$(b_2) \omega_b(a, c) = \omega_b(c, a);$$

$$(b_3) \omega_b(a, c) \leq s[\omega_b(a, e) + \omega_b(e, c)].$$

The pair (Υ, ω_b) is called a b -metric-like space (with a coefficient $s \geq 1$).

Example 1 *Let $\Upsilon = [0, \infty)$. Define the function $\omega_b : \Upsilon^2 \rightarrow [0, \infty)$ by $\omega_b(a, c) = (a + c)^2$. Then, (Υ, ω_b) is a b -metric-like space with constant $s = 2$, but it neither be a b -metric nor a metric-like space (see, [8]).*

For more new examples related with metric-like and b -metric-like spaces, see [14, 15].

Definition 5 [10] Let (Υ, ω_b) be a b -metric-like space and $\{a_n\}$ be a sequence of points on Υ . A point $a \in \Upsilon$ is called the limit of the sequence $\{a_n\}$ if $\lim_{n \rightarrow \infty} \omega_b(a, a_n) = \omega_b(a, a)$, and we say that the sequence $\{a_n\}$ is convergent to a and denote it by $a_n \rightarrow a$ as $n \rightarrow \infty$.

Definition 6 [10] Let (Υ, ω_b) be a b -metric-like space.

- (i) A sequence $\{a_n\}$ is called Cauchy if and only if $\lim_{m, n \rightarrow \infty} \omega_b(a_m, a_n)$ exists and is finite.
- (ii) A b -metric-like space (X, ω_b) is said to be complete if and only if every Cauchy sequence $\{a_n\}$ in Υ converges to $a \in \Upsilon$ such that $\lim_{m, n \rightarrow \infty} \omega_b(a_m, a_n) = \omega_b(a, a) = \lim_{n \rightarrow \infty} \omega_b(a_n, a)$.

Lemma 1 [10] Let $\{c_n\}$ be a sequence in a b -metric-like space (Υ, ω_b) such that

$$\omega_b(c_n, c_{n+1}) \leq \lambda \omega_b(c_{n-1}, c_n),$$

for some $\lambda \in (0, \frac{1}{3})$, and for all $n \in \mathbb{N}$. Then, $\lim_{n \rightarrow \infty} \omega_b(c_m, c_n) = 0$.

Let $\Upsilon \neq \emptyset$. Then, $(\Upsilon, \omega_b, \preceq)$ is called a partially ordered b -metric-like space if ω_b is a b -metric-like space on a partially ordered set (Υ, \preceq) . Further, $(\Upsilon, \omega_b, \preceq)$ is called a regular partially ordered b -metric-like space if it has the following property:

- (i) if a non – decreasing sequence $\{a_n\} \rightarrow a$, then $a_n \preceq a$, for all n ,
- (ii) if a non – increasing sequence $\{c_n\} \rightarrow c$, then $c \preceq c_n$ for all n .

In 1987, Guo and Lakshmikantham [16] first studied the concept of coupled fixed points. Later on, Bhaskar and Lakshmikantham [17] studied monotone property and supported this by providing an application to the existence of periodic boundary value problems. The obtained results were further extended by Lakshmikantham and Ćirić [18] to introducing the notion of a coupled coincidence point and coupled fixed point of mappings in partially ordered metric spaces.

Definition 7 [17] An element $(a, c) \in \Upsilon \times \Upsilon$ is called a coupled fixed point for the mapping $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ if $\Xi(a, c) = a$ and $\Xi(c, a) = c$.

Definition 8 [17] An element $(a, c) \in \Upsilon \times \Upsilon$ is called a coupled coincidence point of the mappings $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ and $\Theta : \Upsilon \rightarrow \Upsilon$ if $\Xi(a, c) = \Theta a$ and $\Xi(c, a) = \Theta c$.

Definition 9 [19] An element $(a, c) \in \Upsilon \times \Upsilon$ is called a coupled coincidence point of mappings $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ if $\Xi(a, c) = \Theta(a, c)$ and $\Xi(c, a) = \Theta(c, a)$.

Example 2 Let $\Xi, \Theta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\Xi(a, c) = ac$ and $\Theta(a, c) = \frac{2}{3}(a + c)$ for all $(a, c) \in \Upsilon \times \Upsilon$. It is clear that $(0, 0)$, $(1, 2)$, and $(2, 1)$ are coupled coincidence points of non-self mappings Ξ and Θ .

Definition 10 [19] Suppose that $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ are two mappings. Ξ is said to be Θ -increasing with respect to \preceq if for all $a, c, l, m \in \Upsilon$, with $\Theta(a, c) \preceq \Theta(l, m)$ we have $\Xi(a, c) \preceq \Xi(l, m)$.

Example 3 Let $\Upsilon = (0, \infty)$ be endowed with the natural ordering of real numbers \leq . Define mappings $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ by $\Xi(a, c) = \ln(a + c)$ and $\Theta(a, c) = a + c$ for all $(a, c) \in \Upsilon \times \Upsilon$. Then, Ξ is Θ -increasing with respect to \leq .

Throughout this paper, we assume that $\Psi = \{\psi : \psi : [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and lower semi-continuous function, $\psi(t) = 0 \Leftrightarrow t = 0\}$.

In 2014, Ansari [20] introduced the concept of C-class functions which cover a large class of contractive conditions.

Definition 11 [20] A mapping $f : [0, \infty)^2 \rightarrow \mathbb{R}$ is called C-class function if it is continuous and satisfies the following conditions:

- (1) $f(\delta, \kappa) \leq \delta$,
- (2) $f(\delta, \kappa) = \delta$ implies that either $\delta = 0$ or $\kappa = 0$ for all $\delta, \kappa \in [0, \infty)$.

We denote C-class functions as \mathbb{C} .

Example 4 The following functions $\Xi : [0, \infty)^2 \rightarrow \mathbb{R}$ are elements of \mathbb{C} , for all $\delta, \kappa \in [0, \infty)$:

- (1) $f(\delta, \kappa) = \delta - \kappa$;
- (2) $f(\delta, \kappa) = \lambda\delta, 0 < \lambda < 1$;
- (3) $f(\delta, \kappa) = \frac{\delta}{(1+\kappa)^r}; r \in (0, \infty)$;
- (4) $f(\delta, \kappa) = \frac{\log(\kappa+a^\delta)}{(1+\kappa)}, a > 1$;
- (5) $f(\delta, 1) = \frac{\ln(1+b^\delta)}{2}, b > e$;
- (6) $f(\delta, \kappa) = \phi(s)$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a upper semi-continuous function such that $\phi(0) = 0$, and $\phi(\kappa) < \kappa$ for $\kappa > 0$;
- (7) $f(\delta, \kappa) = \delta\beta(\delta), \beta : [0, 1) \rightarrow [0, 1)$;
- (8) $f(\delta, \kappa) = \delta - \phi(\delta)$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a upper semi-continuous function such that $\phi(0) = 0$;
- (9) $f(\delta, \kappa) = \delta - \frac{\kappa}{1-\kappa}$;
- (10) $f(\delta, \kappa) = \delta h(\delta, \kappa)$, where $h : [0, 1) \times [0, 1) \rightarrow [0, 1)$ is a continuous function such that $h(\kappa, \delta) < 1$ for all $\kappa, \delta > 0$.

In this paper, we prove some new coincidence coupled point results for the mappings $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ by using the notion of C-class of functions in RPOCbML spaces. We present some examples and an application to find the existence of solution of nonlinear integral equation to support our work. Our contributions in this direction improve and extend several comparable results in the literature for b-metric-like spaces.

Main results

Theorem 1 Let $(\Upsilon, \leq, \omega_b)$ be a RPOCbML space (with a coefficient $s > 1$). Assume that $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ are two mappings such that the following conditions are satisfied:

- (1) $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon \times \Upsilon)$.
- (2) $\Theta(\Upsilon \times \Upsilon)$ is closed.
- (3) Ξ is Θ -increasing with respect to \leq .
- (4) There exist two elements $a_0, c_0 \in \Upsilon$, with $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Theta(c_0, a_0) \geq \Xi(c_0, a_0)$. Suppose that there exist $\psi \in \Psi, 0 < L < 1$ and $f \in \mathbb{C}$ such that:

$$\psi \left(s^\alpha \omega_b(\Xi(a, c), \Xi(l, m)) \right) \leq f \left(\psi \left(\max \{ \omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l)) \} \right), L \max \{ \omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l)) \} \right), \tag{1}$$

for all $a, c, l, m \in \Upsilon, \alpha > 0$ with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(c, a) \leq \Theta(m, l)$ and $\Theta(a, c) \geq \Theta(l, m)$. Then, Ξ and Θ have a coupled coincidence point in Υ .

Proof Let $a_0, c_0 \in \Upsilon$ be an arbitrary with $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Xi(c_0, a_0) \leq \Theta(c_0, a_0)$. Since $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon \times \Upsilon)$, there exists $(a_1, c_1) \in \Upsilon \times \Upsilon$ such that $\Xi(a_0, c_0) = \Theta(a_1, c_1)$ and $\Xi(c_0, a_0) = \Theta(c_1, a_1)$. Continuing this process, we can construct two sequences $\{a_n\}$ and $\{c_n\}$ in Υ such that

$$\Xi(a_n, c_n) = \Theta(a_{n+1}, c_{n+1}), \quad \Xi(c_n, a_n) = \Theta(c_{n+1}, a_{n+1}), \quad \forall n \in \mathbb{N}.$$

Now, we shall prove by induction that for all $n \in \mathbb{N}$, we have

$$\Theta(a_n, c_n) \leq \Theta(a_{n+1}, c_{n+1}) \text{ and } \Theta(c_{n+1}, a_{n+1}) \leq \Theta(c_n, a_n). \tag{2}$$

Since $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Xi(c_0, a_0) \leq \Theta(c_0, a_0)$ and since $\Xi(a_0, c_0) = \Theta(a_1, c_1)$ and $\Xi(c_0, a_0) = \Theta(c_1, a_1)$, we have $\Theta(a_0, c_0) \leq \Theta(a_1, c_1)$ and $\Theta(c_1, a_1) \leq \Theta(c_0, a_0)$. Thus, (2) holds for $n = 0$. Suppose that (2) holds for some fixed $n \in \mathbb{N}$. Since Ξ is Θ -increasing with respect to \leq , we have

$$\Theta(a_{n+1}, c_{n+1}) = \Xi(a_n, c_n) \leq \Xi(a_{n+1}, c_{n+1}) = \Theta(a_{n+2}, c_{n+2}),$$

and

$$\Theta(c_{n+2}, a_{n+2}) = \Xi(c_{n+1}, a_{n+1}) \leq \Xi(c_n, a_n) = \Theta(c_{n+1}, a_{n+1}).$$

Thus, (2) holds for all $n \in \mathbb{N}$. Denote

$$\eta_n = \max \left\{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \right\}.$$

If $\Theta(a_n, c_n) = \Theta(a_{n+1}, c_{n+1})$, then (a_n, c_n) is a coincidence point and the proof is finished. So, we consider $\Theta(a_n, c_n) \neq \Theta(a_{n+1}, c_{n+1})$ for all $n \in \mathbb{N}$, and we claim that

$$\psi(s^\alpha \eta_{n+1}) \leq \psi(\eta_n).$$

Since $\Theta(a_n, c_n) \leq \Theta(a_{n+1}, c_{n+1})$ and $\Theta(c_n, a_n) \geq \Theta(c_{n+1}, a_{n+1})$, using $a = a_n, c = c_n, l = a_{n+1}$, and $m = c_{n+1}$ in (1), we get

$$\begin{aligned} \psi(s^\alpha \omega_b(\Theta(a_{n+1}, c_{n+1}), \Theta(a_{n+2}, c_{n+2}))) &= \psi(s^\alpha \omega_b(\Xi(a_n, c_n), \Xi(a_{n+1}, c_{n+1}))) \\ &\leq f \left(\psi \left(\max \{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \} \right), \right. \\ &\quad \left. L \max \{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \} \right). \end{aligned} \tag{3}$$

Similarly, we can write

$$\begin{aligned} \psi(s^\alpha \omega_b(\Theta(c_{n+2}, a_{n+2}), \Theta(c_{n+1}, a_{n+1}))) &= \psi(s^\alpha \omega_b(\Theta(c_{n+1}, a_{n+1}), \Theta(c_n, a_n))) \\ &\leq f \left(\psi \left(\max \{ \omega_b(\Theta(c_{n+1}, a_{n+1}), \Theta(c_n, a_n)), \omega_b(\Theta(a_{n+1}, c_{n+1}), \Theta(a_n, c_n)) \} \right), \right. \\ &\quad \left. L \max \{ \omega_b(\Theta(c_{n+1}, a_{n+1}), \Theta(c_n, a_n)), \omega_b(\Theta(a_{n+1}, c_{n+1}), \Theta(a_n, c_n)) \} \right). \end{aligned} \tag{4}$$

From (3) and (4), since ψ is non-decreasing, we obtain that

$$\begin{aligned} &\psi \left(\max \{ s^\alpha \omega_b(\Theta(a_{n+1}, c_{n+1}), \Theta(a_{n+2}, c_{n+2})), s^\alpha \omega_b(\Theta(c_{n+2}, a_{n+2}), \Theta(c_{n+1}, a_{n+1})) \} \right), \\ &= \max \{ \psi \left(s^\alpha \omega_b(\Theta(a_{n+1}, c_{n+1}), \Theta(a_{n+2}, c_{n+2})) \right), \psi \left(s^\alpha \omega_b(\Theta(c_{n+2}, a_{n+2}), \Theta(c_{n+1}, a_{n+1})) \right) \}, \\ &\leq f \left(\psi \left(\max \{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \} \right), \right. \\ &\quad \left. L \max \{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \} \right). \end{aligned}$$

This means

$$\psi(s^\alpha \eta_{n+1}) \leq f(\psi(\eta_n), L\eta_n) \leq \psi(\eta_n). \tag{5}$$

Since the function ψ is non-decreasing, so the inequality (5) implies that

$$\eta_{n+1} \leq \frac{1}{s^\alpha} \eta_n.$$

Therefore, by Lemma (1), we have $\{\eta_n\}$ is a Cauchy sequence. Since

$$\eta_n = \max \left\{ \omega_b(\Theta(a_n, c_n), \Theta(a_{n+1}, c_{n+1})), \omega_b(\Theta(c_n, a_n), \Theta(c_{n+1}, a_{n+1})) \right\},$$

then both sequences $\{\Theta(a_n, c_n)\}$ and $\{\Theta(c_n, a_n)\}$ are Cauchy in the complete space (Y, ω_b) , and since $\Xi(Y \times Y) \subseteq \Theta(Y \times Y)$ and $\Theta(Y \times Y)$ is closed, there exist $a, c \in Y$ such that

$$\lim_{n \rightarrow \infty} \Theta(a_n, c_n) = \Theta(a, c) \text{ and } \lim_{n \rightarrow \infty} \Theta(c_n, a_n) = \Theta(c, a).$$

Correspondingly, we have

$$\omega_b(\Theta(a, c), \Theta(a, c)) = \lim_{n \rightarrow \infty} \omega_b(\Theta(a_n, c_n), \Theta(a, c)) = 0, \tag{6}$$

and

$$\omega_b(\Theta(c, a), \Theta(c, a)) = \lim_{n \rightarrow \infty} \omega_b(\Theta(c_n, a_n), \Theta(c, a)) = 0. \tag{7}$$

By the regularity of the space (Y, ω_b, \preceq) , we get

$$\Theta(a_n, c_n) \preceq \Theta(a, c) \text{ and } \Theta(c_n, a_n) \succeq \Theta(c, a).$$

It follows from (1) that

$$\begin{aligned} \psi(s^\alpha \omega_b(\Xi(a_n, c_n), \Xi(a, c))) &\leq f(\psi(\max\{\omega_b(\Theta(a_n, c_n), \Theta(a, c)), \omega_b(\Theta(c_n, a_n), \Theta(c, a))\}), \\ &\quad L \max\{\omega_b(\Theta(a_n, c_n), \Theta(a, c)), \omega_b(\Theta(c_n, a_n), \Theta(c, a))\}). \end{aligned}$$

Passing the limit as $n \rightarrow +\infty$, using (6) and (7), we have

$$\lim_{n \rightarrow \infty} \omega_b(\Xi(a_n, c_n), \Xi(a, c)) = 0 \text{ implies } \Xi(a_n, c_n) = \Xi(a, c). \tag{8}$$

Similarly, using (6) and (7), we obtain

$$\Xi(c_n, a_n) = \Xi(c, a). \tag{9}$$

By triangle inequality in b -metric-like space, one can write

$$\begin{aligned} \omega_b(\Theta(a, c), \Xi(a, c)) &\leq s[\omega_b(\Theta(a, c), \Theta(a_{n+1}, c_{n+1})) + \omega_b(\Theta(a_{n+1}, c_{n+1}), \Xi(a, c))] \\ &= s[\omega_b(\Theta(a, c), \Theta(a_{n+1}, c_{n+1})) + \omega_b(\Xi(a_n, c_n), \Xi(a, c))]. \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using (6) and (8), we have

$$\omega_b(\Theta(a, c), \Xi(a, c)) = 0 \text{ implies } \Theta(a, c) = \Xi(a, c).$$

Similarly, by (7) and (9), we can show that

$$\omega_b(\Theta(c, a), \Xi(c, a)) = 0 \text{ implies } \Theta(c, a) = \Xi(c, a).$$

Thus, Ξ and Θ have a coupled coincidence point. □

To prove the uniqueness of a coupled coincidence point, define the partial ordered in $(Y \times Y, \preceq)$ by for all $(a, c), (l, m) \in Y \times Y$, $(a, c) \preceq (l, m)$ if and only if $\Theta(a, c) \preceq \Theta(l, m)$ and $\Theta(c, a) \succeq \Theta(m, l)$ where $\Theta : Y \times Y \rightarrow Y$ is one-one.

Theorem 2 *In addition to the hypotheses of Theorem 1, suppose that for every $(a, c), (a^*, c^*)$ in $\Upsilon \times \Upsilon$, there exists another (l, m) in $\Upsilon \times \Upsilon$ which is comparable to (a, c) and (a^*, c^*) , then Ξ and Θ have a unique coupled coincidence point.*

Proof From Theorem 1, we know that the set of coupled coincidence points of Ξ and Θ is nonempty. Suppose (a, c) and (a^*, c^*) are coupled coincidence points of Ξ and Θ that is

$$\Xi(a, c) = \Theta(a, c), \quad \Xi(c, a) = \Theta(c, a),$$

and

$$\Xi(a^*, c^*) = \Theta(a^*, c^*), \quad \Xi(c^*, a^*) = \Theta(c^*, a^*).$$

Now we prove that $\Theta(a, c) = \Theta(a^*, c^*)$ and $\Theta(c, a) = \Theta(c^*, a^*)$. By assumption, there exists (l, m) in $\Upsilon \times \Upsilon$ that is comparable to (a, c) and (a^*, c^*) . We define sequences $\{\Theta(l_n, m_n)\}$ and $\{\Theta(m_n, l_n)\}$ as follows:

$$l_0 = l, m_0 = m, \Xi(l_n, m_n) = \Theta(l_{n+1}, m_{n+1}), \Xi(m_n, l_n) = \Theta(m_{n+1}, l_{n+1}), \forall n \in \mathbb{N}.$$

Since (l, m) is comparable to (a, c) , we assume that $(a, c) \leq (l, m) = (l_0, m_0)$. This implies $\Theta(a, c) \leq \Theta(l_0, m_0)$ and $\Theta(c, a) \geq \Theta(m_0, l_0)$. Suppose that $(a, c) \leq (l_n, m_n)$ for some n . We claim that

$$(a, c) \leq (l_{n+1}, m_{n+1}).$$

Since Ξ is Θ increasing, then $\Theta(a, c) \leq \Theta(l_n, m_n)$ implies $\Xi(a, c) \leq \Xi(l_n, m_n)$ and $\Theta(c, a) \geq \Theta(m_n, l_n)$ implies $\Xi(c, a) \geq \Xi(m_n, l_n)$.

Thus,

$$\Theta(a, c) = \Xi(a, c) \leq \Xi(l_n, m_n) = \Theta(l_{n+1}, m_{n+1}),$$

and

$$\Theta(c, a) = \Xi(c, a) \geq \Xi(m_n, l_n) = \Theta(m_{n+1}, l_{n+1}).$$

It follows that

$$(a, c) \leq (l_n, m_n), \forall n. \tag{10}$$

Using (1) and (10), we have

$$\begin{aligned} \psi (s^\alpha \omega_b(\Theta(a, c), \Theta(l_{n+1}, m_{n+1}))) &= \psi (s^\alpha \omega_b(\Xi(a, c), \Xi(l_n, m_n))) \\ &\leq f(\psi (\max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\})), \\ &L \max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\}. \end{aligned} \tag{11}$$

Similarly,

$$\begin{aligned} \psi (s^\alpha \omega_b(\Theta(m_{n+1}, l_{n+1}), \Theta(c, a))) &= \psi (s^\alpha \omega_b(\Xi(m_n, l_n), \Xi(c, a))) \\ &\leq f(\psi (\max\{\omega_b(\Theta(m_n, l_n), \Theta(c, a)), \omega_b(\Theta(l_n, m_n), \Theta(a, c))\})), \\ &L \max\{\omega_b(\Theta(m_n, l_n), \Theta(c, a)), \omega_b(\Theta(l_n, m_n), \Theta(a, c))\}. \end{aligned} \tag{12}$$

Using (11), (12) with the property of f and since ψ is nondecreasing, we have

$$\begin{aligned} \psi (\max\{s^\alpha \omega_b(\Theta(a, c), \Theta(l_{n+1}, m_{n+1})), s^\alpha \omega_b(\Theta(m_{n+1}, l_{n+1}), \Theta(c, a))\}) \\ \leq f(\psi (\max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\})), \\ L \max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\} \\ \leq \psi (\max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\}). \end{aligned} \tag{13}$$

Using the property of ψ in (13), we get

$$\begin{aligned} & \max\{\omega_b(\Theta(a, c), \Theta(l_{n+1}, m_{n+1})), \omega_b(\Theta(m_{n+1}, l_{n+1}), \Theta(c, a))\} \\ & \leq \frac{1}{s^\alpha} \max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\}. \end{aligned}$$

It follows that $\max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\}$ is nonnegative decreasing sequence and

$$\lim_{n \rightarrow \infty} \max\{\omega_b(\Theta(a, c), \Theta(l_n, m_n)), \omega_b(\Theta(c, a), \Theta(m_n, l_n))\} = 0,$$

which implies that

$$\lim_{n \rightarrow \infty} \omega_b(\Theta(a, c), \Theta(l_n, m_n)) = \lim_{n \rightarrow \infty} \omega_b(\Theta(c, a), \Theta(m_n, l_n)) = 0. \tag{14}$$

Repeating the same process, we can prove that

$$\lim_{n \rightarrow \infty} \omega_b(\Theta(a^*, c^*), \Theta(l_n, m_n)) = \lim_{n \rightarrow \infty} \omega_b(\Theta(c^*, a^*), \Theta(m_n, l_n)) = 0. \tag{15}$$

By triangle inequality in b-metric-like space, we have

$$\omega_b(\Theta(a, c), \Theta(a^*, c^*)) \leq s[\omega_b(\Theta(a, c), \Theta(l_n, m_n)) + \omega_b(\Theta(a^*, c^*), \Theta(l_n, m_n))],$$

and

$$\omega_b(\Theta(c, a), \Theta(c^*, a^*)) \leq s[\omega_b(\Theta(c, a), \Theta(m_n, l_n)) + \omega_b(\Theta(c^*, a^*), \Theta(m_n, l_n))].$$

Passing the limit as $n \rightarrow \infty$ into the above inequalities, using (14) and (15), we get $\Theta(a, c) = \Theta(a^*, c^*)$ and $\Theta(c, a) = \Theta(c^*, a^*)$. □

The following examples justify all requirements of the hypotheses of Theorem 1.

Example 5 Let $f(\delta, \kappa) = \tau\delta$, $\Upsilon = [0, \infty)$ endowed with the natural ordering of real numbers. We endow Υ with

$$\omega_b(a, c) = (\max\{a, c\})^2$$

for all $a, c \in \Upsilon$. Then, (Υ, ω_b) is a complete b-metric-like space with a coefficient $s = 2$. Let the sequence $\{a_n\}$ of monotone non-decreasing in Υ such that $\lim_{n \rightarrow \infty} a_n = a \in \Upsilon$, then the sequences (sequences of real numbers)

$$a_1(t) \leq a_2(t) \leq \dots \leq a_n(t) \leq \dots$$

converge also to $a(t)$ for all $t \in [0, \infty)$. So $a_n(t) \leq a(t)$ for all $t \in \Upsilon$, $n \in \mathbb{N}$. Therefore, $a_n \leq a$ for all n . By the same manner, one can show that the monotone non-increasing sequence $\{c_n\}$ in Υ such that $\lim_{n \rightarrow \infty} c_n = c \in \Upsilon$ is a lower bound for all the elements in the sequence, i.e., $c \leq c_n$ for all n . Therefore, the regularity condition hold.

Also, if we define an order relation on Υ as $a \leq c$ for all $a, c \in \Upsilon$, we conclude that Υ is a partially ordered set, so we deduce that $(\Upsilon, \omega_b, \leq)$ is a RPOCbML space.

Define mappings $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ as follows:

$$\Xi(a, c) = \begin{cases} \frac{a-c}{4}, & a \geq c \\ 0, & a < c \end{cases} \quad \text{and} \quad \Theta(a, c) = \begin{cases} a - c, & a \geq c \\ 0, & a < c \end{cases},$$

for all $a, c \in \Upsilon$. It is clear that $\Xi(\Upsilon, \Upsilon) \subseteq \Theta(\Upsilon, \Upsilon)$ and $\Theta(\Upsilon, \Upsilon)$ is closed.

Now, we are going to prove that Ξ is Θ -increasing. Suppose that $(a, c), (l, m) \in \Upsilon \times \Upsilon$ with $\Theta(a, c) \leq \Theta(l, m)$, we state the following cases:

Case 1. If $a < c$, then $\Xi(a, c) = 0 \leq \Theta(a, c)$.

Case 2. If $a \geq c$ and $l \geq m$, then $\Theta(a, c) \leq \Theta(l, m)$ leads to $a - c \leq l - m$, so

$$\frac{a - c}{4} \leq \frac{l - m}{4} \text{ implies } \Xi(a, c) \leq \Xi(l, m).$$

Case 3. If $l < m$, then $\Theta(a, c) \leq \Theta(l, m)$ implies that $0 \leq a - c \leq 0$, hence $a = c$, so $\Xi(a, c) = 0 \leq \Theta(a, c)$. From the above cases, we deduce that Ξ is Θ -increasing.

Next, we prove that there exist two elements $a_o, c_o \in \Upsilon$ with $\Theta(a_o, c_o) \leq \Xi(a_o, c_o)$ and $\Theta(c_o, a_o) \geq \Xi(c_o, a_o)$.

Since we get

$$\Theta(0, \frac{1}{3}) = 0 = \Xi(0, \frac{1}{3}) \text{ and } \Theta(\frac{1}{3}, 0) = \frac{1}{3} > \frac{1}{12} = \Xi(\frac{1}{3}, 0).$$

So the two elements are $0, \frac{1}{3} \in \Upsilon$. Now, define the function $\psi : [0, \infty) \rightarrow [0, \infty)$ as $\psi(\kappa) = \frac{1}{4}\kappa$ for all $\kappa \in [0, \infty)$; it is obvious that $\psi \in \Psi$. Finally, we justify the contraction (1) for all $a, b, l, m \in \Upsilon$, with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(a, c) \geq \Theta(l, m)$ and $\Theta(c, a) \leq \Theta(m, l)$, we get

$$\begin{aligned} \psi(s^\alpha(\Xi(a, c), \Xi(l, m))) &= \frac{1}{2} 2^2 \omega_b(\Xi(a, c), \Xi(l, m)) \\ &= 2(\max\{\Xi(a, c), \Xi(l, m)\})^2 \\ &= \frac{2}{16}(\max\{\Theta(a, c), \Theta(l, m)\})^2 \\ &\leq \frac{1}{4} \omega_b(\Theta(a, c), \Theta(l, m)) \\ &\leq \frac{1}{4} \max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\} \\ &= \frac{1}{2} \psi(\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}) \\ &= m\psi(\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}) \\ &= f\left(\begin{matrix} \psi(\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}) \\ L \max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\} \end{matrix}\right), \end{aligned}$$

where $L \in (0, 1), \alpha = 2 > 0$, and $\tau = \frac{1}{2}$. Hence, the hypothesis (1) is satisfied. Thus, all requirements of the hypotheses of Theorem 1 hold and $(0, 0)$ is a coupled coincidence point of Ξ and Θ and in the same time is a coupled fixed point.

Example 6 Let $f(\delta, \kappa) = \delta - \kappa, \Upsilon = [0, \infty)$ endowed with the natural ordering of real numbers. We endow Υ with

$$\omega_b(a, c) = a^2 + c^2 + |a - c|^2,$$

for all $a, c \in \Upsilon$. Then, $(\Upsilon, \leq, \omega_b)$ is a RPOCbML space with a coefficient $s = 2$ (as in the above example).

Define mappings $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ as follows:

$$\Xi(a, c) = \begin{cases} \frac{1}{8} \ln(1 + \frac{a-c}{2}), & a \geq c \\ 0, & a < c \end{cases} \text{ and } \Theta(a, c) = \begin{cases} \frac{a-c}{2}, & a \geq c \\ 0, & a < c \end{cases},$$

for all $a, b, l, m \in \Upsilon$. It is clear that $\Xi(\Upsilon, \Upsilon) \subseteq \Theta(\Upsilon, \Upsilon)$ and $\Theta(\Upsilon, \Upsilon)$ is closed.

Now, we should show that Ξ is Θ -increasing. Let $(a, c), (l, m) \in \Upsilon \times \Upsilon$ with $\Theta(a, c) \leq \Theta(l, m)$, we state the following cases:

Case 1. If $a < c$, then $\Xi(a, c) = 0 \leq \Theta(a, c)$.

Case 2. If $a \geq c$ and $l \geq m$, then $\Theta(a, c) \leq \Theta(l, m)$, leads to $\frac{a-c}{2} \leq \frac{l-m}{2}$, so

$$\frac{\ln(1 + \frac{a-c}{2})}{16} \leq \frac{\ln(1 + \frac{l-m}{2})}{16} \text{ implies } \Xi(a, c) \leq \Xi(l, m).$$

Case 3. If $l < m$, then $\Theta(a, c) \leq \Theta(l, m)$ implies that

$$0 \leq \frac{a-b}{2} \leq 0 \text{ implies } a - b = 0 \text{ implies } a = b,$$

so $\Xi(a, c) = 0 \leq \Theta(a, c)$. From the three cases above, we deduce that Ξ is Θ -increasing.

Next, as in Example 5, we prove that there exist two elements $a_o, c_o \in \Upsilon$ with $\Theta(a_o, c_o) \leq \Xi(a_o, c_o)$ and $\Theta(a_o, c_o) \geq \Theta(c_o, a_o)$. So, we get

$$\Theta(0, 1) = 0 = \Xi(0, 1) \text{ and } \Theta(1, 0) = \frac{1}{2} > \frac{1}{8} \ln\left(\frac{3}{2}\right) = \Xi(1, 0).$$

So the two elements are $0, 1 \in \Upsilon$. Now, define the function $\psi : [0, \infty) \rightarrow [0, \infty)$ as $\psi(\kappa) = \frac{1}{8}\kappa$ for all $\kappa \in [0, \infty)$; it is obvious that $\psi \in \Psi$. Finally, we verify the contraction (1) for all $a, c, l, m \in \Upsilon$, with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(a, c) \geq \Theta(l, m)$ and $\Theta(c, a) \leq \Theta(m, l)$. Since $\ln(1 + \kappa) \leq \kappa$ for all $\kappa \in [0, \infty)$, then we have

$$\begin{aligned} & \psi\left(s^\alpha (\Xi(a, c), \Xi(l, m))\right) = \\ &= \frac{1}{8} 2^2 \omega_b (\Xi(a, c), \Xi(l, m)) \\ &= \frac{1}{128} \left[\left(\ln\left(1 + \frac{a-c}{2}\right) \right)^2 + \left(\ln\left(1 + \frac{l-m}{2}\right) \right)^2 + \left| \ln\left(1 + \frac{a-c}{2}\right) - \ln\left(1 + \frac{l-m}{2}\right) \right|^2 \right] \\ &\leq \frac{1}{128} \left[\left(\frac{a-c}{2} \right)^2 + \left(\frac{l-m}{2} \right)^2 + \left| \frac{a-c}{2} - \frac{l-m}{2} \right|^2 \right] \\ &\leq \frac{1}{128} \omega_b (\Theta(a, c), \Theta(l, m)) \\ &\leq \frac{1}{128} \max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \} \\ &= \left(\frac{1}{8} - \frac{15}{128} \right) \max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \} \\ &= \psi (\max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \}) \\ &\quad - L \max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \} \\ &= f \left(\begin{array}{l} \psi (\max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \}) \\ L \max \{ \omega_b (\Theta(a, c), \Theta(l, m)), \omega_b (\Theta(c, a), \Theta(m, l)) \} \end{array} \right), \end{aligned}$$

where $L = \frac{15}{128} < 1$ and $\alpha = 2 > 0$. Hence, the condition (1) is satisfied. Thus, all requirements of the hypotheses of Theorem 1 are verified and $(0, 0)$ is a coupled coincidence point of Ξ and Θ , and in the same time is a coupled fixed point.

Consequences of the main results

If we put $f(\delta, \kappa) = \delta - \kappa$ in Theorem 1, we get the following corollary:

Corollary 1 Let $(\Upsilon, \leq, \omega_b)$ be a RPOCbML space (with parameter $s > 1$). Assume that $\Xi, \Theta : \Upsilon \times \Upsilon \rightarrow \Upsilon$ are two mappings such that the following conditions are satisfied:

- (1) $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon \times \Upsilon)$.
- (2) $\Theta(\Upsilon \times \Upsilon)$ is closed.
- (3) Ξ is Θ -increasing with respect to \leq .
- (4) There exist two elements $a_0, c_0 \in \Upsilon$, with $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Theta(c_0, a_0) \geq \Xi(c_0, a_0)$. Suppose that there exist $\psi \in \Psi, 0 < L < 1$ such that:

$$\psi (s^\alpha \omega(\Xi(a, c), \Xi(l, m))) \leq \psi (\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\} - L \max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}),$$

for all $a, c, l, m \in \Upsilon$, $\alpha > 0$ with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(c, a) \leq \Theta(m, l)$ and $\Theta(a, c) \geq \Theta(l, m)$. Then, Ξ and Θ have a coupled coincidence point in Υ .

If we choose $f(\delta, \kappa) = k\delta$, $0 \leq k < 1$ in Theorem 1, we get the following result:

Corollary 2 Let (X, \leq, ω_b) be a RPOCbML space (with coefficient $s > 1$). Assume that $F, G : X \times X \rightarrow X$ are two mappings such that the following conditions are realized:

- (1) $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon \times \Upsilon)$.
- (2) $\Theta(\Upsilon \times \Upsilon)$ is closed.
- (3) Ξ is Θ -increasing with respect to \leq .
- (4) There exist two elements $a_0, c_0 \in \Upsilon$, with $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Theta(c_0, a_0) \geq \Xi(c_0, a_0)$. Suppose that there exist $\psi \in \Psi$ such that:

$$\psi (s^\alpha \omega(\Xi(a, c), \Xi(l, m))) \leq k\psi (\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}),$$

for all $a, c, l, m \in \Upsilon$, $\alpha > 0$ with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(c, a) \leq \Theta(m, l)$ and $\Theta(a, c) \geq \Theta(l, m)$. Then, Ξ and Θ have a coupled coincidence point in Υ .

If we take $\psi(t) = t$ in Theorem 1, we get the following consequence:

Corollary 3 Let (X, \leq, ω_b) be a RPOCbML space (with coefficient $s > 1$). Assume that $F, G : X \times X \rightarrow X$ are two mappings such that the following conditions hold:

- (1) $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon \times \Upsilon)$.
- (2) $\Theta(\Upsilon \times \Upsilon)$ is closed.
- (3) Ξ is Θ -increasing with respect to \leq .
- (4) There exist two elements $a_0, c_0 \in \Upsilon$, with $\Theta(a_0, c_0) \leq \Xi(a_0, c_0)$ and $\Theta(c_0, a_0) \geq \Xi(c_0, a_0)$. Suppose that $0 < L < 1$ such that:

$$s^\alpha \omega(\Xi(a, c), \Xi(l, m)) \leq f(\max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}, L \max\{\omega_b(\Theta(a, c), \Theta(l, m)), \omega_b(\Theta(c, a), \Theta(m, l))\}),$$

for all $a, c, l, m \in \Upsilon$, $\alpha > 0$ with $\Theta(a, c) \leq \Theta(l, m)$ and $\Theta(c, a) \geq \Theta(m, l)$ or $\Theta(c, a) \leq \Theta(m, l)$ and $\Theta(a, c) \geq \Theta(l, m)$. Then, Ξ and Θ have a coupled coincidence point in Υ .

Definition 12 [21] Let (Υ, \leq) be a partially ordered set, $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ and $\Theta : \Upsilon \rightarrow \Upsilon$. We say that Ξ is Θ -increasing with respect to \leq if for any $a, c \in \Upsilon$

$$\Theta a_1 \leq \Theta a_2 \text{ implies } \Xi(a_1, c) \leq \Xi(a_2, c) \text{ and } \Theta c_1 \leq \Theta c_2 \text{ implies } \Xi(a, c_1) \leq \Xi(a, c_2).$$

Taking in to account the above definition, we obtain the following result:

Corollary 4 Let $(\Upsilon, \leq, \omega_b)$ be a RPOCbML space (with parameter $s > 1$). Assume that $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ and $\Theta : \Upsilon \rightarrow \Upsilon$ are two mappings such that the following conditions are verified:

- (1) $\Xi(\Upsilon \times \Upsilon) \subseteq \Theta(\Upsilon)$.
- (2) $\Theta(\Upsilon)$ is closed.
- (3) Ξ is Θ -increasing with respect to \leq .

(4) There exist two elements $a_0, c_0 \in \Upsilon$ such that $\Theta a_0 \preceq \Xi(a_0, c_0)$ and $\theta c_0 \succeq \Xi(c_0, a_0)$. Suppose that there exist $\psi \in \Psi, 0 < L < 1$ and $f \in C$ such that:

$$\psi \left(s^\alpha \omega_b(\Xi(a, c), \Xi(l, m)) \right) \leq f \left(\psi \left(\max\{\omega_b(\Theta a, \Theta l), \omega_b(\Theta c, \Theta m)\} \right), L \max\{\omega_b(\Theta a, \Theta l), \omega_b(\Theta c, \Theta m)\} \right),$$

for all $a, c, l, m \in \Upsilon, \alpha > 0$ with $\Theta a \preceq \Theta l$ and $\Theta c \succeq \Theta m$ or $\Theta c \preceq \Theta m$ and $\Theta a \succeq \Theta l$. Then, Ξ and Θ have a coupled coincidence point in Υ .

Definition 13 [17] Let (Υ, \preceq) be a partially ordered set, $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$. We say that Ξ is increasing with respect to \preceq if for any $a, c \in \Upsilon$

$$a_1 \preceq a_2 \text{ implies } \Xi(a_1, c) \preceq \Xi(a_2, c) \text{ and } c_1 \preceq c_2 \text{ implies } \Xi(a, c_1) \preceq \Xi(a, c_2).$$

According to the above definition, we have the following corollary:

Corollary 5 Let $(\Upsilon, \preceq, \omega_b)$ be a RPOCbML space (with parameter $s > 1$). Assume that $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ is mapping such that the following conditions are justified:

- (1) Ξ is increasing with respect to \preceq .
- (2) There exist two elements $a_0, c_0 \in X$ such that $a_0 \preceq \Xi(a_0, c_0)$ and $c_0 \succeq \Xi(c_0, a_0)$. Suppose that there exist $\psi \in \Psi, 0 < L < 1$ and $f \in C$ such that:

$$\psi \left(s^\alpha \omega_b(\Xi(a, c), \Xi(l, m)) \right) \leq f \left(\psi \left(\max\{\omega_b(a, l), \omega_b(c, m)\} \right), L \max\{\omega_b(a, l), \omega_b(c, m)\} \right),$$

for all $a, c, l, m \in \Upsilon, \alpha > 0$ with $a \preceq l$ and $c \succeq m$ or $c \preceq m$ and $a \succeq l$. Then, Ξ has a coupled fixed point in Υ .

Supportive application

Recently fixed point theory has expanded to its importance in many applications including finding analytical solutions for differential and integral equations (see [22–24] and references therein). In fact, this section is the most important part of the paper, where we shall study the existence of a unique solution to a non-linear integral equation using the theoretical results presented in the previous section.

Let Π denote the class functions $\pi : [0, \infty) \rightarrow [0, \infty)$ such that π is increasing and there exists $\psi \in \Psi, 0 < L < 1$, and $f \in \mathbb{C}$ such that $\pi(\kappa) = \frac{1}{4}f(\psi(\kappa), L\kappa)$ for all $\kappa \in [0, \infty)$. Suppose the following problem:

$$a(t) = \int_p^q (K_1(t, \rho) + K_2(t, \rho)) \left(\frac{g(\rho, a(\rho)) + h(\rho, a(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, a(\rho))} \right) d\rho, \tag{16}$$

for all $t \in [p, q]$. Suppose that K_1, K_2, g, h , satisfies the following hypotheses:

- (i) For all $t, \rho \in [p, q], K_1(t, \rho) \geq 0$ and $K_2(t, \rho) \geq 0$;
- (ii) For all $a, c \in \mathbb{R}$ such that $a \geq c$, there exist σ, ς such that

$$\begin{aligned} 0 &\leq g(\rho, a) + g(\rho, c) \leq \sigma \pi(a + c), \\ 0 &\leq h(\rho, a) + h(\rho, c) \leq \varsigma \pi(a + c); \end{aligned}$$

(iii) We get

$$\max\{\sigma^2, \varsigma^2\} \left\{ \sup_{t \in [a,b]} \int_p^q (K_1(t, \rho) + K_2(t, \rho)) d\rho \right\}^2 \leq 1;$$

(iv) For all $t \in [p, q]$, there exist continuous functions $\varpi, \xi : [p, q] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \varpi(t) &\leq \int_p^q K_1(t, \rho) \left(\frac{g(\rho, \varpi(\rho)) + h(\rho, \xi(\rho))}{1 + g(\rho, \xi(\rho)) + h(\rho, \varpi(\rho))} \right) d\rho \\ &\quad + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, \xi(\rho)) + h(\rho, \varpi(\rho))}{1 + g(\rho, \varpi(\rho)) + h(\rho, \xi(\rho))} \right) d\rho, \end{aligned}$$

and

$$\begin{aligned} \xi(t) &\geq \int_p^q K_1(t, \rho) \left(\frac{g(\rho, \xi(\rho)) + h(\rho, \varpi(\rho))}{1 + g(\rho, \varpi(\rho)) + h(\rho, \xi(\rho))} \right) d\rho \\ &\quad + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, \varpi(\rho)) + h(\rho, \xi(\rho))}{1 + g(\rho, \xi(\rho)) + h(\rho, \varpi(\rho))} \right) d\rho. \end{aligned}$$

Now, we formulate the theorem concerning the existence of a unique solution of a nonlinear integral equation (16) as follows:

Theorem 3 *Suppose that the problem (16) with $K_1, K_2 \in C([p, q] \times [p, q], \mathbb{R})$ and $g, h \in C([p, q] \times \mathbb{R} \times \mathbb{R})$, then the problem (16) has a solution in $C([p, q], \mathbb{R})$, provided that hypotheses (i)–(iv) are verified.*

Proof Assume that $\Upsilon = C([a, b], \mathbb{R})$ is the set of real continuous functions on $[p, q]$ equipped with

$$\omega_b(a, c) = \sup_{t \in [p,q]} (|a(t) + c(t)|)^2 \text{ for all } a, c \in \Upsilon.$$

Thus, the pair (Υ, ω_b) is a complete b -metric-like space with a coefficient $s = 2$, and $(\Upsilon, \omega_b, \preceq)$ is a RPOCbML space if $a \preceq c$ whenever $a(t) \leq c(t)$ for all $t \in [p, q]$. Let the sequence of monotone non-decreasing $\{a_n\}$ in Υ such that $\lim_{n \rightarrow \infty} a_n = a \in \Upsilon$, then the sequences (sequences of real numbers)

$$a(t) \leq a_2(t) \leq \dots \leq a_n(t) \leq \dots$$

converge also to $a(t)$ for all $t \in [p, q]$. So $a_n(t) \leq a(t)$ for all $t \in [p, q]$, $n \in \mathbb{N}$. Therefore, $a_n \leq a$ for all n . By the same manner, one can show that the monotone non-increasing sequence $\{c_n\}$ in Υ such that $\lim_{n \rightarrow \infty} c_n = c \in \Upsilon$ is a lower bound for all the elements in the sequence, i.e., $c \leq c_n$ for all n . Therefore, the regularity condition hold.

Also, if we define an order relation on $\Upsilon \times \Upsilon$ as $a \preceq l, c \succeq m$ for all $a, c, l, m \in \Upsilon$, we conclude that $\Upsilon \times \Upsilon$ is a partially ordered set.

Define the mapping $\Xi : \Upsilon \times \Upsilon \rightarrow \Upsilon$ by

$$\begin{aligned} \Xi(a, c)(t) &= \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a(\rho)) + h(\rho, c(\rho))}{1 + g(\rho, c(\rho)) + h(\rho, a(\rho))} \right) d\rho \\ &\quad + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c(\rho)) + h(\rho, a(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, c(\rho))} \right) d\rho, \end{aligned}$$

for all $t \in [p, q]$. It is clear that if the mapping Ξ has a coupled fixed point in $\Upsilon = C([p, q], \mathbb{R})$, then it is a solution of the problem (16).

Now, we shall prove that the mapping Ξ is increasing with $a_1 \leq a_2$, so, $a_1(t) \leq a_2(t)$ for all $t \in [p, q]$; we can write

$$\begin{aligned} &\Xi(a_1, c)(t) - \Xi(a_2, c)(t) \\ &= \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a_1(\rho)) + h(\rho, c(\rho))}{1 + g(\rho, c(\rho)) + h(\rho, a_1(\rho))} \right) d\rho + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c(\rho)) + h(\rho, a_1(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, c(\rho))} \right) d\rho \\ &\quad - \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a_2(\rho)) + h(\rho, c(\rho))}{1 + g(\rho, c(\rho)) + h(\rho, a_2(\rho))} \right) d\rho - \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c(\rho)) + h(\rho, a_2(\rho))}{1 + g(\rho, a_2(\rho)) + h(\rho, c(\rho))} \right) d\rho \\ &\leq \int_p^q K_1(t, \rho) (g(\rho, a_1(\rho)) + h(\rho, c(\rho))) d\rho + \int_p^q K_2(t, \rho) (g(\rho, c(\rho)) + h(\rho, a_1(\rho))) d\rho \\ &\quad - \int_p^q K_1(t, \rho) (g(\rho, a_2(\rho)) + h(\rho, c(\rho))) d\rho - \int_p^q K_2(t, \rho) (g(\rho, c(\rho)) + h(\rho, a_2(\rho))) d\rho \\ &= \int_p^q K_1(t, \rho) (g(\rho, a_1(\rho)) - g(\rho, a_2(\rho))) d\rho + \int_p^q K_2(t, \rho) (h(\rho, a_1(\rho)) - h(\rho, a_2(\rho))) d\rho \leq 0. \end{aligned}$$

Therefore, $\Xi(a_1, c)(t) \leq \Xi(a_2, c)(t)$ for all $t \in [p, q]$. Hence, $\Xi(a_1, c) \leq \Xi(a_2, c)$. Similarly, if $c_1 \leq c_2$, so $c_1(t) \leq c_2(t)$ for all $t \in [p, q]$, we have

$$\begin{aligned} &\Xi(a, c_1)(t) - \Xi(a, c_2)(t) \\ &= \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a(\rho)) + h(\rho, c_1(\rho))}{1 + g(\rho, c_1(\rho)) + h(\rho, a(\rho))} \right) d\rho + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c_1(\rho)) + h(\rho, a(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, c_1(\rho))} \right) d\rho \\ &\quad - \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a(\rho)) + h(\rho, c_2(\rho))}{1 + g(\rho, c_2(\rho)) + h(\rho, a(\rho))} \right) d\rho - \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c_2(\rho)) + h(\rho, a(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, c_2(\rho))} \right) d\rho \\ &\leq \int_p^q K_1(t, \rho) (g(\rho, a(\rho)) + h(\rho, c_1(\rho))) d\rho + \int_p^q K_2(t, \rho) (g(\rho, c_1(\rho)) + h(\rho, a(\rho))) d\rho \\ &\quad - \int_p^q K_1(t, \rho) (g(\rho, a(\rho)) + h(\rho, c_2(\rho))) d\rho - \int_p^q K_2(t, \rho) (g(\rho, c_2(\rho)) + h(\rho, a(\rho))) d\rho \\ &= \int_p^q K_1(t, \rho) (h(\rho, c_1(\rho)) - h(\rho, c_2(\rho))) d\rho + \int_p^q K_2(t, \rho) (g(\rho, c_1(\rho)) - g(\rho, c_2(\rho))) d\rho \leq 0. \end{aligned}$$

Therefore, $\Xi(a, c_1)(t) \leq \Xi(a, c_2)(t)$ for all $t \in [p, q]$. Hence, $\Xi(a, c_1) \leq \Xi(a, c_2)$. From the above results, we deduce that the mapping Ξ is increasing with respect to the variables a, c .

Finally, we illustrate that the contraction condition of Corollary 5 hold, for all $a, c, l, m \in \Upsilon$ such that $a \leq l, m \leq c$,

$$\begin{aligned} & \omega_b (\Xi(a, c), \Xi(l, m)) \\ &= \sup_{t \in [p, q]} (|\Xi(a, c)(t) + \Xi(l, m)(t)|)^2 \\ &= \left\{ \sup_{t \in [p, q]} \left| \int_p^q K_1(t, \rho) \left(\frac{g(\rho, a(\rho)) + h(\rho, c(\rho))}{1 + g(\rho, c(\rho)) + h(\rho, a(\rho))} \right) d\rho + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, c(\rho)) + h(\rho, a(\rho))}{1 + g(\rho, a(\rho)) + h(\rho, c(\rho))} \right) d\rho \right. \right. \\ & \quad \left. \left. + \int_p^q K_1(t, \rho) \left(\frac{g(\rho, l(\rho)) + h(\rho, m(\rho))}{1 + g(\rho, m(\rho)) + h(\rho, l(\rho))} \right) d\rho + \int_p^q K_2(t, \rho) \left(\frac{g(\rho, m(\rho)) + h(\rho, l(\rho))}{1 + g(\rho, l(\rho)) + h(\rho, m(\rho))} \right) d\rho \right| \right\}^2 \\ &\leq \left\{ \sup_{t \in [p, q]} \left| \left(\int_p^q K_1(t, \rho) (g(\rho, a(\rho)) + h(\rho, c(\rho))) d\rho \right) \right. \right. \\ & \quad \left. \left. + \left(\int_p^q K_2(t, \rho) (g(\rho, c(\rho)) + h(\rho, a(\rho))) d\rho \right) \right| \right. \\ & \quad \left. + \left(\int_p^q K_1(t, \rho) (g(\rho, l(\rho)) + h(\rho, m(\rho))) d\rho \right) \right. \\ & \quad \left. + \left(\int_p^q K_2(t, \rho) (g(\rho, m(\rho)) + h(\rho, l(\rho))) d\rho \right) \right| \right\}^2 \\ &= \left\{ \sup_{t \in [p, q]} \left| \left(\int_p^q K_1(t, \rho) [(g(\rho, a(\rho)) + g(\rho, l(\rho))) + (h(\rho, c(\rho)) + h(\rho, m(\rho)))] d\rho \right) \right. \right. \\ & \quad \left. \left. + \left(\int_p^q K_2(t, \rho) [(g(\rho, c(\rho)) + g(\rho, m(\rho))) + (h(\rho, a(\rho)) + h(\rho, l(\rho)))] d\rho \right) \right| \right\}^2. \end{aligned}$$

Applying condition (ii), we have

$$\begin{aligned} & \omega_b (\Xi(a, c), \Xi(l, m)) \\ &\leq \left\{ \sup_{t \in [p, q]} \left(\int_p^q K_1(t, \rho) [\sigma \pi(a(\rho) + l(\rho)) + \varsigma \pi(c(\rho) + m(\rho))] d\rho \right) \right. \\ & \quad \left. + \left(\int_p^q K_2(t, \rho) [\sigma \pi(c(\rho) + m(\rho)) + \varsigma \pi(a(\rho) + l(\rho))] d\rho \right) \right\}^2 \\ &\leq \max\{\sigma^2, \varsigma^2\} \left\{ \sup_{t \in [p, q]} \left(\int_p^q [K_1(t, \rho) + K_2(t, \rho)] [\pi(|a(\rho) + l(\rho)|) + \pi(|c(\rho) + m(\rho)|)] d\rho \right) \right\}^2. \tag{17} \end{aligned}$$

By the definition of π and the distance ω_b , we can write

$$\pi |a(\rho) + l(\rho)|^2 \leq \pi \omega_b(a, l) \text{ and } \pi |c(\rho) + m(\rho)|^2 \leq \pi \omega_b(c, m) \text{ for all } \rho \in [p, q]. \tag{18}$$

It follows from (17), (18), and condition (iii) that

$$\begin{aligned} & \omega_b (\Xi(a, c), \Xi(l, m)) \\ &\leq \max\{\sigma^2, \varsigma^2\} \times [\pi^2 \omega_b(a, l) + \pi^2 \omega_b(c, m)] \times \left\{ \sup_{t \in [p, q]} \int_p^q (K_1(t, \rho) + K_2(t, \rho)) d\rho \right\}^2 \\ &\leq \pi^2 \omega_b(a, l) + \pi^2 \omega_b(c, m) \\ &\leq \pi^2 (\omega_b(a, l) + \omega_b(c, m)) \\ &\leq 2\pi^2 (\max\{\omega_b(a, l), \omega_b(c, m)\}) \text{ since } i + j \leq 2 \max\{i, j\} \text{ for all } i, j \in \mathbb{R} \\ &\leq 2 \times \frac{1}{16} f(\psi(\max\{\omega_b(a, l), \omega_b(c, m)\}), L \max\{\omega_b(a, l), \omega_b(c, m)\}) \\ &= \frac{1}{8} f(\psi(\max\{\omega_b(a, l), \omega_b(c, m)\}), L \max\{\omega_b(a, l), \omega_b(c, m)\}). \end{aligned}$$

Thus, for $a \leq l, m \leq c$, we obtain that

$$2^3 \omega_b (\Xi(a, c), \Xi(l, m)) \leq f (\psi (\max \{\omega_b(a, l), \omega_b(c, m)\}), L \max \{\omega_b(a, l), \omega_b(c, m)\}),$$

taking $\psi \in \Psi$ such that $\psi(\kappa) = \kappa$, for all $\kappa \in [0, \infty]$ with $s = 2$ and $\alpha = 3$, we have

$$\psi(s^\alpha \omega_b (\Xi(a, c), \Xi(l, m))) \leq f (\psi (\max \{\omega_b(a, l), \omega_b(c, m)\}), L \max \{\omega_b(a, l), \omega_b(c, m)\}).$$

Also, condition (iv) leads to $\varpi(t) \leq \Xi(\varpi, \xi)(t)$ and $\xi(t) \geq \Xi(\xi, \varpi)(t)$ for all $t \in [p, q]$, that is $\varpi \preceq \Xi(\varpi, \xi)$ and $\xi \succeq \Xi(\xi, \varpi)$. Therefore, all the hypothesis requirements of Corollary 5 are satisfied. Then, the mapping Ξ has a coupled fixed point which is a solution of the problem (16) in $\Upsilon = C([p, q], \mathbb{R})$. \square

Conclusion

Recently, the technique of coincidence point has become one of the important issues that contribute to finding the analytical solution of integral equations, which in turn facilitates the numerical solution. Therefore, many researchers are interested in this trend so that the differential equation can be converted to their integral counterparts and then easily solved. This paper is the same line, where we discussed some common coincidence points for nonlinear mappings under the class of a RPOCbML space; also, we proceed with two examples to support theoretical results and analytical solution for the nonlinear integral equation illustrated.

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HAH wrote the title, abstract, examples, application, and conclusion and fixed many language errors. DMA wrote the theoretical work, introduction, and references. RAR fixed some spelling errors and gave some important sentences to support the paper. All authors read and approved the final manuscript.

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