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The T–R {Y} power series family of probability distributions

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Abstract

A new family of univariate probability distributions called the $T - R$ {Y} power series family of probability distributions is introduced in this paper by compounding the $T - R$ {Y} family of distributions and the power series family of discrete distributions. A treatment of the general mathematical properties of the new family is carried out and some sub-families of the new family are specified to depict the broadness of the new family. The maximum likelihood method of parameter estimation is suggested for the estimation of the parameters of the new family of distributions. A special member of the new family called the Gumbel–Weibull–{logistic}–Poisson (GUWELOP) distribution is defined and found to exhibit both unimodal and bimodal shapes. The GUWELOG distribution is further applied to a real multi-modal data set to buttress its applicability.

Keywords: $T - R${Y} family, Power series family, Continuous distribution, Discrete distribution, Maximum Likelihood estimation

Mathematics Subject Classification: 62B15, 60E05, 62F10, 62N05

Introduction

Within the last two centuries, various methods for generating continuous univariate distributions have been put forward in the literature. These methods include the method based on differential equations (Pearson [1]; Burr [2]), method based on transformation (Johnson [3]), method based on quantiles (Tukey [4]; Aldeni et al. [5]), method for generating skewed distributions (Azzalini [6]), method of addition of parameter(s) and generalization (Mudholkar and Srivastava [7]; Marshall and Olkin [8]; Shaw and Buckley [9]), method of compounding the continuous univariate distributions and the discrete univariate distributions (Adamidis and Loukas [10]), method based on generators (Eugene et al. [11]; Jones [12]; Cordeiro and de Castro [13]), method based on the composition of densities (Cooray and Ananda [14]) and the Transformed–Transformer method (Alzaatreh et al. [15]; Alzaatreh et al. [16]). Researchers are also encouraged to see AL-Hussaini and Abdel-Hamid [17] for a survey on the generation of distribution functions.

The transformed–transformer method previously called the T–X family of distributions (Alzaatreh [15]) and later renamed the T–R{Y} family of distributions (Alzaatreh et al. [16]) has been thought of as the largest family of univariate distributions, in that it includes several families of univariate distributions as special cases. Alzaatreh et al. [16] defined the T−R{Y} system using the following arguments: Suppose T, R, and Y are random variables with respective cumulative distribution function (cdf) $F_T(x) = P(T \le x)$, $F_R(x) = P(R \le x)$ and $F_Y(x) = P(Y \le x)$. Let the corresponding quantile functions be $Q_T(p)$, $Q_R(p)$ and $Q_Y(p)$, where the quantile function is defined as $Q_W(p) = \inf\{w : F_W(w) \ge p\}$, $0 < p < 1$. Suppose the corresponding densities of T, R and Y exist and denote them by $f_T(x)$, $f_R(x)$ and $f_Y(x)$. Assume that $T\epsilon(a, b)$ and $Y\epsilon(c, d)$for $-\infty \le a < b \le \infty$ and$-\infty \le c < d \le \infty$,then the T−R{Y} family of distributions was defined by the cdf

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P[T \le Q_Y(F_R(x))] = F_T(Q_Y(F_R(x))), x\in\mathbb{R}. \tag{1}$$

The corresponding probability density function (pdf) of the cdf in (1) was given by

$$f_X(x) = f_R(x) \times \frac{f_T(Q_Y(F_R(x)))}{f_Y(Q_Y(F_R(x)))}, x\in\mathbb{R}. \tag{2}$$

The discrete counterpart of univariate probability distributions has also received some attention over the years in the literature. One of the most common families of discrete univariate distributions is the power series family of discrete univariate distributions (Kosambi [18]; Noack [19]; Patil [20]; Patil [21]) defined by the probability mass function (pmf)

$$P(N = n) = \frac{a_n\theta^n}{C(\theta)}, n = 1, 2, \ldots \tag{3}$$

where $a_n \ge 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ and $\theta > 0$ is such that $C(\theta)$ is finite and its first, second and third derivatives are defined and shown by $C'(\theta)$, $C''(\theta)$, and $C'''(\theta)$. Observe that the pmf in (3) is truncated at zero and could be generalized to a zero-inflated one (Patil, [21]). In Table 1, some members of the power series family of distributions (truncated at zero) defined by (3) such as the Poisson, geometric, binomial and logarithmic distributions are presented alongside their respective a_n, $C(\theta)$, $C'(\theta)$, $C''(\theta)$, and $C'''(\theta)$.

In this paper, the compounding of the T–R $\{Y\}$ family of univariate distributions and the power series family of discrete univariate distributions is carried out. We shall present how the new family is constructed, examine the general mathematical properties of the new family, show how parameters of the new family can be estimated using the maximum likelihood method as well as define and apply a special member of the new family to a real data set.

Table 1 Useful quantities for some power series distributions

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C'''(\theta)$	$C^{-1}(\theta)$	parameter space
Binomial	$\binom{m}{n}$	$(1+\theta)^m - 1$	$m(1+\theta)^{m-1}$	$\frac{m(m-1)}{(1+\theta)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1+\theta)^{3-m}}$	$(\theta-1)^{1}/m-1$	$\theta \in (0, 1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$6(1-\theta)^{-4}$	$\theta(1+\theta)^{-1}$	$\theta \in (0, 1)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$1-e^{-\theta}$	$\theta \in (0, 1)$
Poisson	$(n!)^{-1}$	e^{θ}-1	e^{θ}	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta \in (0, \infty)$

Source: Morais and Barreto-Souza [22]
Note: In the table, m is the number of trials or replicas in the binomial experiment

Construction of the *T–R* {*Y*} power series family of distributions

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (iid) random variables constituting a sample of size n from the $T–R$ $\{Y\}$ family of distributions as defined in (1). Let $X_{(1)}, X_{(2)}, \ldots, X_{(N)}$ be the corresponding order statistic of the random sample. From the theory of order statistics, the cdf of first order statistic $X_{(1)}$ for a given $N = n$ is expressed as

$$Z_{X_{(1)}|N=n}(x) = 1-\prod_{i=1}^{n}[1-F_{T_i}(Q_Y(F_R(x)))] = 1-[1-F_T(Q_Y(F_R(x)))]^n.$$

Suppose N is a discrete random variable and follows the power series distribution in (3), the marginal cdf of $X_{(1)}$ can be written as

$$F_{\mathrm{T-R\ \{Y\}-PS}}(x) = \sum_{n=1}^{\infty} P(N=n) Z_{X_{(1)}|N=n}(x) = 1-\frac{C[\theta(1-F_T(Q_Y(F_R(x))))]}{C(\theta)}.$$

Thus, the cdf of the $T–R$ $\{Y\}$–power series ($T–R$ $\{Y\}$–PS) family of distributions is given by

$$F_{\mathrm{T\text{-}R\{Y\}-PS}}(x) = 1-\frac{C[\theta(1-F_T(Q_Y(F_R(x))))]}{C(\theta)}, x \in \mathbb{R}. \tag{4}$$

A physical interpretation of the family of models in (4) is as follows: consider that the failure of a system, device, product, or component occurs due to the presence of an unknown number, say N, of initial defects of the same kind, which can be identifiable only after causing the failure and repaired perfectly. If X_i denotes the time to the failure of the device due to the ith defect, for $i \geq 1$, such that each X_i follows the $T–R$ $\{Y\}$ distribution in (1), suppose N is discrete and follows a power series distribution in (3), then the distribution of the random variable $X_{(1)}$ which is the time of first failure is the distribution in (4).

The pdf corresponding to (4) is obtained by differentiating (4) w.r.t x and it is given by

$$f_{\mathrm{T\text{-}R\{Y\}-PS}}(x) = \frac{\theta C'[\theta(1-F_T(Q_Y(F_R(x))))] f_X(x)}{C(\theta)}, x \in \mathbb{R}. \tag{5}$$

The survival and hazard functions of the $T–R$ $\{Y\}$–PS family of distributions are given respectively by

$$S_{\mathrm{T\text{-}R\{Y\}-PS}}(x) = \frac{C[\theta(1-F_T(Q_Y(F_R(x))))]}{C(\theta)}, x \in \mathbb{R}, \tag{6}$$

$$h_{\mathrm{T\text{-}R\{Y\}-PS}}(x) = \frac{\theta C'[\theta(1-F_T(Q_Y(F_R(x))))] f_X(x)}{C[\theta(1-F_T(Q_Y(F_R(x))))]}, x \in \mathbb{R}. \tag{7}$$

Some sub-families of the $T–R\{Y\}$—PS family of distributions namely: $T–R$ $\{Y\}$—binomial ($T–R\{Y\}–B$) distribution, $T–R$ $\{Y\}$–Poisson ($T–R\{Y\}–P$) distribution, $T–R$ $\{Y\}$—geometric ($T–R\{Y\}–G$) distribution and the $T–R$ $\{Y\}$–logarithmic ($T–R\{Y\}–L$) distribution are defined in Table 2 by their cdfs. In Table 3, five standardized distributions of the random variable Y are presented alongside their various quantile functions $Q_Y(p)$

and the corresponding support of the random variable T which is needed to make (1) a valid cdf. These standardized distributions include the standard exponential, logistic, extreme value, log logistic, and uniform distributions. The use of standardized distributions is to reduce the number of parameters in the $T–R\{Y\}$–PS distributions. For practical purposes and when highly necessary, these standardized distributions can be replaced with their non-standardized versions.

In Tables 4, 5, 6, and 7, different $T–R\{Y\}–B$, $T–R\{Y\}–G$, $T–R\{Y\}–L$, and $T–R\{Y\}–P$ distributions are presented respectively for different choices of $Q_Y(p)$ in Table 3.

General mathematical properties of the *T–R* {*Y*} power series family of distributions

Some useful statistical properties of the new family are presented. We begin by looking at some limiting distributions as contained in Propositions 1 and 2.

Limiting distributions and some useful representations

Proposition 1:

The $T–R\{Y\}$ distribution defined by (1) is a limiting case of the $T-R\{Y\}-$PS family of distributions defined in (4) when $\theta \to 0^+$.

Proof:

Applying $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$, one readily obtains

$$F_{T-R\{Y\}-PS}(x) = 1-\frac{\sum_{n=1}^{\infty} a_n[\theta(1-F_T(Q_Y(F_R(x))))]^n}{\sum_{n=1}^{\infty} a_n\theta^n}.$$

Considering $\theta \to 0^+$, we have

$$\lim_{\theta\to 0^+} F_{\mathrm{T-R\{Y\}}-PS}(x) = 1-\lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty} a_n[\theta(1-F_T(Q_Y(F_R(x))))]^n}{\sum_{n=1}^{\infty} a_n\theta^n}.$$

Evaluating using standard procedure gives

$$\lim_{\theta\to 0^+} F_{\mathrm{T-R\{Y\}}-PS}(x) = 1-\frac{a_1(1-F_T(Q_Y(F_R(x))))}{a_1} = F_T(Q_Y(F_R(x))),$$

which is the cdf of the $T-R$ $\{Y\}$ distribution defined by (1).

Table 2 Some sub-families of the *T–R*{*Y*}–PS family of distributions

Distributions	cdf
$T–R\{Y\}$–B	$1-\frac{(1+\theta(1-F_T(Q_Y(F_R(x)))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$
$T–R\{Y\}$–G	$\frac{F_T(Q_Y(F_R(x)))}{1-\theta(1-F_T(Q_Y(F_R(x))))}, x\in\mathbb{R}.$
$T–R\{Y\}–L$	$1-\frac{\log(1-\theta(1-F_T(Q_Y(F_R(x)))))}{\log(1-\theta)}, x\in\mathbb{R}.$
$T–R\{Y\}–P$	$1-\frac{e^{\theta(1-F_T(Q_Y(F_R(x))))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$

Table 3 Some distributions of Y with corresponding $Q_Y(p)$ and support of T

Standardized distributions of the random variable Y	The quantile function $Q_Y(p)$	Support of T
Exponential	$-\log(1-p)$	$0 < T < \infty$
Logistic	$\log(p/(1-p))$	$-\infty < T < \infty$
Extreme value	$\log(-\log(1-p))$	$-\infty < T < \infty$
log logistic	$p/(1-p)$	$0 < T < \infty$
uniform	p	$0 < T < 1$

Proposition 2:

For $Q_Y(F_R(x)) = x$ and $\theta \to 0^+$, the $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}-PS$ family of distributions defined in (4) reduces to the distribution of the random variable T.

Proof:

The proof follows directly and explicitly from substituting x for $Q_Y(F_R(x))$ in (1) and the proof of Proposition 1.

Proposition 3:

The pdf of the $T-R\{Y\}-\mathrm{PS}$ family of distributions can be expressed as linear combination of density of the first order statistic of the $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}$ distribution as

$$f_{\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}-PS}(x) = \sum_{n=1}^{\infty} P(N=n) f_{X_{x_{(1)}}}(x;n),$$

where $f_{X_{x_{(1)}}}(x;n)$ is the pdf of $X_{(1)} = \min\{X_i\}_{i=1}^n$

Proof:

Observe that $C'(\theta) = \sum_{i=1}^{\infty} n a_n \theta^{n-1}$. Using (5), one readily obtains

$$f_{T-R\{Y\}-PS}(x) = \sum_{n=1}^{\infty} \frac{a_n \theta^n}{C(\theta)} n f_X(x)[1 - F_T(Q_Y(F_R(x)))]^{n-1},$$

and $f_{X_{x_{(1)}}}(x;n) = n f_X(x)[1 - F_T(Q_Y(F_R(x))),]^{n-1}$. Hence, the proof.

Quantiles and moments

The quantile function and moments of a probability distribution provide the theoretical base upon which many statistical properties of a distribution are assessed with. The quantile function in particular is very useful in Monte Carlo simulations since it helps in producing simulated random variates for any distribution, especially when it is in closed form.

Table 4 Different T–$R\{Y\}$–B distributions

Distributions	cdf
T–R\{exponential\}–B	$1-\frac{(1+\theta(1-F_T(-\log(1-F_R(x)))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$
T–R\{logistic\}–B	$1-\frac{(1+\theta(1-F_T(\log(F_R(x)/(1-F_R(x))))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$
T–R\{extreme value\}–B	$1-\frac{(1+\theta(1-F_T(\log(-\log(1-F_R(x))))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$
T–R\{log logistic\}–B	$1-\frac{(1+\theta(1-F_T(F_R(x)/(1-F_R(x)))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$
T–R\{uniform\}–B	$1-\frac{(1+\theta(1-F_T(F_R(x))))^m-1}{(1+\theta)^m-1}, x\in\mathbb{R}.$

Table 5 Different *T*–*R*{*Y*}–*G* distributions

Distributions	cdf
T–*R*{exponential}–*G*	$\frac{F_T(-\log(1-F_R(x)))}{1-\theta(1-F_T(-\log(1-F_R(x))))}, x\in\mathbb{R}.$
T–*R*{logistic}–G	$\frac{F_T(\log(F_R(x)/(1-F_R(x))))}{1-\theta(1-F_T(\log(F_R(x)/(1-F_R(x)))))}, x\in\mathbb{R}.$
T–*R*{extreme value}–*G*	$\frac{F_T(\log(-\log(1-F_R(x))))}{1-\theta(1-F_T(\log(-\log(1-F_R(x)))))}, x\in\mathbb{R}.$
T–*R*{log logistic}–*G*	$\frac{F_T(F_R(x)/(1-F_R(x)))}{1-\theta(1-F_T(F_R(x)/(1-F_R(x))))}, x\in\mathbb{R}.$
T–*R*{uniform}–*G*	$\frac{F_T(F_R(x))}{1-\theta(1-F_T(F_R(x)))}, x\in\mathbb{R}.$

Theorem 1:

The quantile function $Q(p)$ of the $T-R\{Y\}-PS$ family of distributions is given by

$$Q(p) = Q_R\left\{F_Y\left[Q_T\left(1-\frac{C^{-1}((1-p)C(\theta))}{\theta}\right)\right]\right\},\ \ 0 < p < 1, \tag{8}$$

where $C^{-1}(.)$ is the inverse of $C(.)$

Proof:

The result in (8) is obtained by solving the equation $F_{\mathrm{T-R\{Y\}}-PS}(Q(p)) = p$ for $Q(p)$.

Corollary 1:

Random samples can be simulated from the $T-R\{Y\}-\mathrm{PS}$ family of distributions by making use of the relation

$$X = Q_R\left\{F_Y\left[Q_T\left(1-\frac{C^{-1}((1-U)C(\theta))}{\theta}\right)\right]\right\},\ \ 0 < U < 1, \tag{9}$$

where X is a $T-R\{Y\}-\mathrm{PS}$ random variable and U, a uniform random variable on the interval (0, 1).

Proof:

The proof follows by substituting U for p in (8), where U is a uniform random variable on the interval (0, 1).

An expression for the rth non-central moments of the $T-R\{Y\}-\mathrm{PS}$ family of distributions random variable follows from Proposition 3. The rth non-central moments of the $T-R\{Y\}-\mathrm{PS}$ family of distributions random variable X is given by

Table 6 Different *T*–*R*{*Y*}–*L* distributions

Distributions	cdf
T–*R*{exponential}–*L*	$1-\frac{\log(1-\theta(1-F_T(-\log(1-F_R(x)))))}{\log(1-\theta)}, x\in\mathbb{R}.$
T–*R*{logistic}–*L*	$1-\frac{\log(1-\theta(1-F_T(\log(F_R(x)/(1-F_R(x))))))}{\log(1-\theta)}, x\in\mathbb{R}.$
T–*R*{extreme value}–*L*	$1-\frac{\log(1-\theta(1-F_T(\log(-\log(1-F_R(x))))))}{\log(1-\theta)}, x\in\mathbb{R}.$
T–*R*{log logistic}–*L*	$1-\frac{\log(1-\theta(1-F_T(F_R(x)/(1-F_R(x)))))}{\log(1-\theta)}, x\in\mathbb{R}.$
T–*R*{uniform}–*L*	$1-\frac{\log(1-\theta(1-F_T(F_R(x))))}{\log(1-\theta)}, x\in\mathbb{R}.$

Table 7 Different T–$R\{Y\}$–P distributions

Distributions	cdf
T–R\{exponential\}–P	$1-\frac{e^{\theta(1-F_T(-\log(1-F_R(x))))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$
T–R\{logistic\}–P	$1-\frac{e^{\theta(1-F_T(\log(F_R(x)/(1-F_R(x)))))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$
T–R\{extreme value\}–P	$1-\frac{e^{\theta(1-F_T(\log(-\log(1-F_R(x)))))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$
T–R\{log logistic\}–P	$1-\frac{e^{\theta(1-F_T(F_R(x)/(1-F_R(x))))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$
T–R\{uniform\}–P	$1-\frac{e^{\theta(1-F_T(F_R(x)))}-1}{e^{\theta}-1}, x\in\mathbb{R}.$

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f_{T-\mathrm{R}\{\mathrm{Y}\}-PS}(x)dx = \sum_{n=1}^{\infty} P(N=n)E\left(X_{(1)}^r\right), \tag{10}$$

where $E(X_{(1)}^r)$ is the rth non-central moment of the first order statistic of a T–$R\{Y\}$ random variable. Thus the rth non-central moments of the $T-R\{Y\}-$PS family of distributions can be expressed as a linear combination of the rth non-central moments of the first order statistics of the $T-R\{Y\}$ distribution.

The moment generating function (mgf) of the $T-R\{Y\}-$PS family of distributions is defined by

$$M_X(t) = E\left(e^{tX}\right).$$

Using Proposition 3, the mgf can be expressed as

$$M_X(t) = \sum_{n=1}^{\infty} P(N=n) M_{X_{(1)}}(t). \tag{11}$$

Thus the mgf of the $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}-PS$ family of distributions can be expressed as a linear combination of the mgf of the first order statistics of the $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}$ distribution.

Order statistics

Order statistics are among the most essential tools in non-parametric statistics and inference. Their importance is highly visible in the problems of estimation and hypotheses tests in a variety of ways. Their moments play an important role in quality control testing and reliability, where an analyst needs to predict the failure of future components or items based on the times of a few observed early failures. These predictors are most of the time based on moments of order statistics.

Theorem 2:

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from the $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}-PS$ family of distributions and suppose $X_{1:m} < X_{2:m} < \ldots < X_{m:m}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic can be expressed as

$$f_{\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}-PS_{k:m}}(x) = \frac{1}{B(k, m-k+1)} \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \delta_{r,n,m,k,j} f_{X_{x_{(1)}}}(x; n+m+j-k+r+1), \tag{12}$$

where $B(.,.)$ is the complete beta function.

$$\delta_{r,n,m,k,j} = \binom{k-1}{j} \frac{(-1)^j (r+1)\theta^{m+j-k+n+r+1} a_1^{m+j-k+1} b_r d_{m+j-k,n}}{[m+j-k+n+r+1](C(\theta))^{m+j-k+1}},$$

$$d_{m+j-k,0} = 1,$$

$$d_{m+j-k,t} = t^{-1}\sum_{n=1}^{t}[n(m+j-k+1)-t]b_n d_{m+j-k,t-n}, t \geq 1,$$

$$b_0 = 1, \qquad b_r = a_{r+1}/a_1 \text{ for } r = 1, 2, 3, ...,$$

$$b_0 = 1, \qquad b_n = a_{n+1}/a_1 \text{ for } n = 1, 2, 3, ...,$$

and $f_{X_{x_{(1)}}}(x; n+m+j-k+r+1)$ denote the pdf of $X_{(1)} = \min\{X_i\}_{i=1}^{n+m+j-k+r+1}$.

Proof:

From definition, the pdf of the kth order statistic of the $T-R\{Y\}-$PS family of distributions can be written as

$$f_{\mathrm{T-R\{Y\}-PS}_{k:m}}(x) = \frac{1}{B(k, m-k+1)} f_{\mathrm{T-R\{Y\}}-PS}(x)\left[F_{\mathrm{T-R\{Y\}}-PS}(x)\right]^{k-1}\left[1-F_{\mathrm{T-R\{Y\}}-PS}(x)\right]^{m-k}. \quad (13)$$

Using the binomial expansion formula, one readily obtains

$$\begin{aligned}\left[F_{\mathrm{T-R\{Y\}}-PS}(x)\right]^{k-1} &= \left[1-\left(1-F_{\mathrm{T-R\{Y\}}-PS}(x)\right)\right]^{k-1}\\ &= \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left[1-F_{\mathrm{T-R\{Y\}}-PS}(x)\right]^j.\end{aligned}$$

Substituting into (13) gives

$$f_{\mathrm{T-R\{Y\}}-PS_{k:m}}(x) = \frac{1}{B(k, m-k+1)} f_{\mathrm{T-R\{Y\}}-PS}(x)\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left[1-F_{\mathrm{T-R\{Y\}}-PS}(x)\right]^{m+j-k}. \quad (14)$$

Substituting (4) and (5) into (14) gives

$$f_{\mathrm{T-R\{Y\}}-PS_{k:m}}(x) = \frac{\theta C'[\theta(1-F_T(Q_Y(F_R(x))))]f_X(x)}{B(k, m-k+1)C(\theta)}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left[\frac{C[\theta(1-F_T(Q_Y(F_R(x))))]}{C(\theta)}\right]^{m+j-k}. \quad (15)$$

Now consider the term

$$(C[\theta(1-F_T(Q_Y(F_R(x))))])^{m+j-k} = \left[\sum_{n=1}^{\infty}a_n\theta^n(1-F_T(Q_Y(F_R(x))))^n\right]^{m+j-k}$$

$$= a_1^{m+j-k}\theta^{m+j-k}(1-F_T(Q_Y(F_R(x))))^{m+j-k}\left[\sum_{n=0}^{\infty}b_n\theta^n(1-F_T(Q_Y(F_R(x))))^n\right]^{m+j-k}$$

where $b_0 = 1$, $b_n = a_{n+1}/a_1$ for $n = 1, 2, 3, ...$

Using the identity

$$\left(\sum_{n=0}^{\infty}b_n z^n\right)^p = \sum_{n=0}^{\infty}d_{p,n}z^n,$$

(see. Gradshteyn and Ryzhik [23]) for a positive integer $m+j-k$, one can write

$$(C[\theta(1-F_T(Q_Y(F_R(x))))])^{m+j-k} = a_1^{m+j-k}\theta^{m+j-k}(1-F_T(Q_Y(F_R(x))))^{m+j-k}\sum_{n=0}^{\infty}d_{m+j-k,n}\theta^n(1-F_T(Q_Y(F_R(x))))^n.$$

Consequently,

$$(C[\theta(1-F_T(Q_Y(F_R(x))))])^{m+j-k} = a_1^{m+j-k}\sum_{n=0}^{\infty} d_{m+j-k,n}\theta^{m+j-k+n}(1-F_T(Q_Y(F_R(x))))^{m+j-k+n} \tag{16}$$

where $d_{m+j-k,\ 0} = 1$ and the coefficients for $t \geq 1$ can be obtained from the recurrence equation

$$d_{m+j-k,t} = t^{-1}\sum_{n=1}^{t}[n(m+j-k+1)-t]b_n d_{m+j-k,t-n}.$$

An expression for $C'[\theta(1-F_T(Q_Y(F_R(x))))]$ can also be defined. In particular,

$$C'[\theta(1-F_T(Q_Y(F_R(x))))] = \sum_{r=1}^{\infty} r a_r \theta^{r-1}(1-F_T(Q_Y(F_R(x))))^{r-1}.$$

Thus,

$$C'[\theta(1-F_T(Q_Y(F_R(x))))] = a_1\sum_{r=0}^{\infty}(r+1)b_r\theta^r(1-F_T(Q_Y(F_R(x))))^r, \tag{17}$$

where $b_0 = 1,\ b_r = a_{r+1}/a_1$ for $r = 1, 2, 3, \ldots$ Inserting (16) and (17) in (15) gives

$$f_{\mathrm{T-R\{Y\}}-PS_{k:m}}(x) = \frac{1}{B(k,m-k+1)}\sum_{j=0}^{k-1}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\delta_{r,n,m,k,j}[m+j-k+n+r+1]f_X(x)(1-F_T(Q_Y(F_R(x))))^{[m+j-k+n+r+1]-1},$$

hence

$$f_{\mathrm{T-R\{Y\}}-PS_{k:m}}(x) = \frac{1}{B(k,m-k+1)}\sum_{j=0}^{k-1}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\delta_{r,n,m,k,j}f_{X_{x_{(1)}}}(x;n+m+j-k+r+1),$$

where

$$\delta_{r,n,m,k,j} = \binom{k-1}{j}\frac{(-1)^j(r+1)\theta^{m+j-k+n+r+1}a_1^{m+j-k+1}b_r d_{m+j-k,n}}{[m+j-k+n+r+1](C(\theta))^{m+j-k+1}},$$

and

$f_{X_{x_{(1)}}}(x;n+m+j-k+r+1)$ denote the pdf of $X_{(1)} = \min\{X_i\}_{i=1}^{n+m+j-k+r+1}$.

One readily observes that the pdf of the $T-R\{\mathrm{Y}\}-\mathrm{PS}$ family order statistics is an infinite linear combination of the density of $X_{(1)} = \min\{X_i\}_{i=1}^{n+m+j-k+r+1}$, where the quantities $\delta_{r,\ n,\ m,\ k,\ j}$ depend only on the power series family.

The sth moment of the $T-R\{\mathrm{Y}\}-\mathrm{PS}$ family kth order statistics is given as

$$E\left(X_{k:m}^s\right) = \int_{\mathbb{R}} x_{k:m}^s\ f_{\mathrm{T-R\{Y\}}-PS_{k:m}}(x_{k:m})dx.$$

Thus,

$$E\left(X_{k:m}^s\right) = \frac{1}{B(k,m-k+1)}\sum_{j=0}^{k-1}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{q=0}^{m+j-k+n+r}\delta_{r,n,m,k,j}\delta_{r,n,m,k,j,q}$$
$$\times\int_{\mathbb{R}} x_{k:m}^s f_X(x_{k:m})(F_T(Q_Y(F_R(x_{k:m}))))^q dx, \tag{18}$$

where

$$\delta_{r,n,m,k,j,q} = (-1)^q \binom{m+j-k+n+r}{q} [m+j-k+n+r+1].$$

A characterization for the new family

Following a dual concept in statistical mechanics, Shannon [24] introduced the probabilistic definition of entropy. The Shannon entropy which is sometimes referred to as a measure of uncertainty plays an essential role in information theory. To measure randomness or uncertainty, the entropy of a random variable comes handy since it can be defined in terms of its probability distribution. Suppose X is a continuous random variable with density function f. Then, the Shannon entropy of X is defined by

$$\mathbb{H}_{Sh}(f) = -\int_{\mathbb{R}} f \log f dx. \tag{19}$$

Another powerful method often employed in the field of probability and statistics and closely related to the Shannon entropy is the "maximum entropy method" pioneered by Jaynes [25]. The method considers a family of density functions

$$\mathbb{F} = \left\{ f : E_f(T_i(X)) = \alpha_i, i = 0, \dots, m \right\},$$

where $T_1(X), \dots, T_m(X)$ are absolutely integrable functions with respect to f, and $T_0(X) = \alpha_0 = 1$. In the continuous case, the maximum entropy principle suggests deriving the unknown density function of the random variable X by the model that maximizes the Shannon entropy (19) subject to the information constraints defined in the family $\mathbb{F}$ (see. Shore and Johnson [26]). The maximum entropy method has been used for the characterization of several standard probability distributions; see for example, Zografos and Balakrishnan [27].

The maximum entropy distribution is the density of the family $\mathbb{F}$, denoted f^{ME}, obtained as the solution of the optimization problem

$$f^{ME} = \arg \max_{f\epsilon\mathbb{F}} \mathbb{H}_{Sh}.$$

As demonstrated by Jaynes [25], the maximum entropy distribution f^{ME} determined by the constrained maximization problem depicted above "is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have" To provide a maximum entropy characterization for the $T-R\{Y\}-$PS family, a derivation of important constraints is undertaken.

Proposition 4:

If X is a random variable with density (5) and Z follows a $\mathrm{T}-\mathrm{R}\{\mathrm{Y}\}$ distribution with density given by (2), the following constraints hold

$$C1\ \ E\left\{ \log C'[\theta(1-F_T(Q_Y(F_R(X))))] \right\} = \frac{\theta}{C(\theta)} E\left\{ C'[\theta(1-F_T(Q_Y(F_R(Z))))] \log C'[\theta(1-F_T(Q_Y(F_R(Z))))] \right\},$$

$$C2\ \ E\{ \log f(X)\} = \frac{\theta}{C(\theta)} E\left\{ \log f(Z) C'[\theta(1-F_T(Q_Y(F_R(Z))))] \right\}.$$

Proof:

The proof is trivial and hence it is omitted.

Theorem 3:

The density function $f_{\mathrm{T-R\{Y\}}-PS}(.)$ given in (5) for the random variable X following the $T-R\{Y\}-\mathrm{PS}$ family of distributions, is the unique solution of the optimization problem

$$f_{\mathrm{T-R\{Y\}}-PS} = \arg \max_{h\in\mathbb{F}} \mathbb{H}_{Sh}(h)$$

under the constraints $C1$ and $C2$ given in Proposition 4.

Proof:

Suppose $\nu(.)$ is a pdf which satisfies the constraints $C1$ and $C2$. The Kullback-Leibler divergence between the densities ν and $f_{\mathrm{T-R\{Y\}}-PS}$ is

$$D\left(\nu, f_{\mathrm{T-R\{Y\}}-PS}\right) = \int_{\mathbb{R}} \nu \log\left(\frac{\nu}{f_{\mathrm{T-R\{Y\}}-PS}}\right) dx.$$

Following Cover and Thomas [28], one obtains

$$\begin{aligned} 0 \le D\left(\nu, f_{\mathrm{T-R\{Y\}}-PS}\right) &= \int_{\mathbb{R}} \nu \log \nu dx - \int_{\mathbb{R}} \nu \log f_{\mathrm{T-R\{Y\}}-PS} dx \\ &= -\mathbb{H}_{Sh}(\nu) - \int_{\mathbb{R}} \nu \log f_{\mathrm{T-R\{Y\}}-PS} dx. \end{aligned}$$

Let Z have the pdf given by (2). From the definition of $f_{\mathrm{T-R\{Y\}}-PS}$ and based on the constraints $C1$ and $C2$, the following result holds:

$$\int_{\mathbb{R}} \nu \log f_{\mathrm{T-R\{Y\}}-PS} dx = \int_{\mathbb{R}} \frac{\theta}{C(\theta)} C'[\theta(1-F_T(Q_Y(F_R(z))))] f(z) \log\left\{\frac{\theta}{C(\theta)} C'[\theta(1-F_T(Q_Y(F_R(z))))] f(z)\right\} dz$$

Since the density ν satisfies the constraints $C1$ and $C2$.

$$\begin{aligned} \int_{\mathbb{R}} \nu \log f_{\mathrm{T\text{-}R\{Y\}}-PS} dx &= \frac{\theta}{C(\theta)} \int_{\mathbb{R}} C'[\theta(1-F_T(Q_Y(F_R(z))))] f(z) \left\{ \log\theta + \log\left\{C'[\theta(1-F_T(Q_Y(F_R(z))))] f(z)\right\} - \log C(\theta)\right\} dz \\ &= \log\theta - \log C(\theta) + \frac{\theta}{C(\theta)} E\left\{C'[\theta(1-F_T(Q_Y(F_R(Z))))] \log C'[\theta(1-F_T(Q_Y(F_R(Z))))]\right\} \\ &\quad + \frac{\theta}{C(\theta)} E\left\{ \log f(Z) C'[\theta(1-F_T(Q_Y(F_R(Z))))]\right\} = -\mathbb{H}_{Sh}\left(f_{\mathrm{T\text{-}R\{Y\}}-PS}\right) \end{aligned} \tag{20}$$

Thus,

$$0 \le \mathbb{H}_{Sh}\left(f_{\mathrm{T-R\{Y\}}-PS}\right) - \mathbb{H}_{Sh}(\nu),$$

hence,

$$\mathbb{H}_{Sh}(\nu) \le \mathbb{H}_{Sh}\left(f_{\mathrm{T-R\{Y\}}-PS}\right),$$

with equality if and only if $\nu(x) = f_{\mathrm{T-R\{Y\}}-PS}(x)$ for all x except for a null measure set. This proves Theorem 3.

Corollary 2:

The Shannon entropy of the $T-R\{Y\}-\mathrm{PS}$ family of distributions is given by

$$\begin{aligned} \mathrm{H}_{sh} = \left(f_{\mathrm{T\text{-}R\{Y\}}-PS}\right) = \log C(\theta) - \log\theta - \frac{\theta}{C(\theta)} E\left\{C'[\theta(1-F_T(Q_Y(F_R(Z))))] \log C'[\theta(1-F_T(Q_Y(F_R(Z))))]\right\} \\ - \frac{\theta}{C(\theta)} E\left\{ \log f(Z) C'[\theta(1-F_T(Q_Y(F_R(Z))))]\right\}. \end{aligned} \tag{21}$$

Proof:

The result follows from (20).

The mode of the family

The mode(s) of the $T-R\{Y\}-\mathrm{PS}$ family of distributions can be obtained as the solution of the equation

$$f'_{\mathrm{T-R\ \{Y\}-PS}}(x) = 0$$

for x. It follows that the mode(s) of a $T-R\{Y\}-\mathrm{PS}$ distribution can be obtained by solving for x in the equation

$$\left[C'[\theta(1-F_T(Q_Y(F_R(x))))]f'_X(x)-\theta C'[\theta(1-F_T(Q_Y(F_R(x))))](f_X(x))^2\right] = 0. \tag{22}$$

Mean deviations of the family

The dispersion and the spread in a population from the center are often measured by the deviation from the mean, and the deviation from the median. The mean absolute deviation about the mean, $D(\mu)$, and the mean absolute deviation about the median, $D(M)$, for the new family are defined as

$$D(\mu) = \int_{-\infty}^{\infty} |x-\mu| f_{\mathrm{T-R\{Y\}}-PS}(x)\ dx,$$

and

$$D(M) = \int_{-\infty}^{\infty} |x-M| f_{\mathrm{T-R\{Y\}}-PS}(x)\ dx,$$

respectively, where $\mu = E(X)$ and $M = Q(0.5)$. Consequently,

$$D(\mu) = \int_{-\infty}^{\infty} |x-\mu| f_{\mathrm{T-R\{Y\}}-PS}(x)dx = \int_{-\infty}^{\mu} (\mu-x) f_{\mathrm{T-R\{Y\}}-PS}(x)dx + \int_{\mu}^{\infty} (x-\mu) f_{\mathrm{T-R\{Y\}}-PS}(x)dx.$$

Thus,

$$D(\mu) = 2\mu F_{\mathrm{T-R\{Y\}}-PS}(\mu)-2\mu + 2\int_{\mu}^{\infty} x f_{\mathrm{T-R\{Y\}}-PS}(x)dx. \tag{23}$$

Also,

$$D(M) = \int_{-\infty}^{\infty} |x-M| f_{\mathrm{T-R\{Y\}}-PS}(x)dx = \int_{-\infty}^{M} (M-x) f_{\mathrm{T-R\{Y\}}-PS}(x)dx + \int_{M}^{\infty} (x-M) f_{\mathrm{T-R\{Y\}}-PS}(x)dx$$

Thus,

$$D(M) = -\mu + 2\int_{M}^{\infty} x f_{\mathrm{T-R\{Y\}}-PS}(x)dx. \tag{24}$$

Remark: Many results obtained so far can be determined numerically by employing any symbolic computing software such as MATLAB, MATHEMATICA, and R. The infinity limit in the sums can be substituted by a large number for applied purposes.

Maximum likelihood estimation of the parameters of the new family

Suppose ξ is a $p \times 1$ vector containing all the parameters of the $T-R\{Y\}$ distribution, for a complete random sample $x_1, x_2, \ldots, x_n$ of size n from the $T-R\{Y\}$ – PS family, the total log-likelihood function is given by

$$\ell = n \log(\theta) - n \log(C(\theta)) + \sum_{i=1}^{n} \log\left(C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right) + \sum_{i=1}^{n} \log(f_X(x_i;\xi)). \tag{25}$$

Let $\Theta = (\theta\ \xi)^T$ be the unknown parameter vector of the $T-R\{Y\}$ – PS family, the associated score function is given by

$$\boldsymbol{U}(\Theta) = \left(\frac{\partial \ell}{\partial \theta}\ \frac{\partial \ell}{\partial \xi}\right)^T,$$

where $\frac{\partial \ell}{\partial \theta}$ and $\frac{\partial \ell}{\partial \xi}$ are given by

$$U_\theta = \frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \frac{n\, C'(\theta)}{C(\theta)} + \sum_{i=1}^{n} \frac{\partial\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\theta}{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]},$$

$$U_{\xi_k} = \frac{\partial \ell}{\partial \xi_k} = \sum_{i=1}^{n} \frac{\partial\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\xi_k}{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]} + \sum_{i=1}^{n} \frac{\partial(f_X(x_i;\xi))/\partial\xi_k}{f_X(x_i;\xi)}.$$

The maximum likelihood estimate of Θ, $\hat{\Theta}$, can be obtained by solving the non-linear systems of equations, $\boldsymbol{U}(\Theta) = 0$. Since the resulting systems of equations are not in closed form, the solutions can be found numerically using some specialized numerical iterative scheme such as the Newton-Raphson type algorithms, which can be implemented on several computing software like R, SAS, MATHEMATICA, and MATLAB.

For interval estimation of the parameters of the $T-R\{Y\}$ – PS family, one would require the Fisher information matrix (FIM) given by the $(1+p) \times (1+p)$ symmetric matrix

$$\boldsymbol{I}(\Theta) = -E_\Theta \begin{pmatrix} U_{\theta\theta} & | & U_{\theta\xi}^T \\ -- & -- & -- \\ U_{\theta\xi} & | & U_{\xi\xi} \end{pmatrix},$$

where p is the number of parameter(s) in the $T-R\{\mathrm{Y}\}$ distribution and

$$U_{\theta\theta} = -\frac{n}{\theta^2} - n\left\{\frac{C(\theta)C''(\theta) - [C'(\theta)]^2}{[C(\theta)]^2}\right\} + \sum_{i=1}^{n} \frac{\partial^2\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\theta^2}{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]} - \sum_{i=1}^{n} \frac{\left(\partial\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\theta\right)^2}{\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}^2},$$

$$U_{\theta\xi_k} = \sum_{i=1}^{n} \frac{\partial^2\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\theta\partial\xi_k}{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]} - \sum_{i=1}^{n} \frac{\partial\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\xi_k \partial\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}/\partial\theta}{\left\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\right\}^2},$$

$$U_{\xi_k\xi_l} = \sum_{i=1}^{n} \frac{\partial^2\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\}/\partial\xi_k\partial\xi_l}{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]}$$

$$-\sum_{i=1}^{n} \frac{\partial\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\}/\partial\xi_k\partial\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\}/\partial\xi_l}{\{C'[\theta(1-F_T(Q_Y(F_R(x_i;\xi))))]\}^2}$$
$$+\sum_{i=1}^{n}\frac{\partial^2(f_X(x_i;\xi))/\partial\xi_k\partial\xi_l}{f_X(x_i;\xi)} - \sum_{i=1}^{n}\frac{\partial(f_X(x_i;\xi))/\partial\xi_k\partial(f_X(x_i;\xi))/\partial\xi_l}{(f_X(x_i;\xi))^2}.$$

The total FIM, $\boldsymbol{I}(\Theta)$, can be approximated by

$$\boldsymbol{J}(\hat{\Theta}) \approx \left[-\frac{\partial^2\ell}{\partial\Theta_i\partial\Theta_j}\bigg|_{\Theta=\hat{\Theta}}\right]_{(1+p)\times(1+p)}.$$

For real data, $\boldsymbol{J}(\hat{\Theta})$ is obtained after the maximum likelihood estimate of Θ is gotten, which implies the convergence of the iterative numerical procedure involved in finding such estimate.

Given that $\hat{\Theta}$ is the maximum likelihood estimate of Θ and under the conditions that are fulfilled for the parameters Θ in the interior of the parameter space but not on the boundary, it follows that $\sqrt{n}(\hat{\Theta}-\Theta)\xrightarrow{d}N_{1+p}(\mathbf{0},\boldsymbol{I}^{-1}(\Theta))$, where $\boldsymbol{I}^{-1}(\Theta)$ is the inverse of the expected FIM. The asymptotic behavior is still valid if $\boldsymbol{I}^{-1}(\Theta)$ is replaced by $\boldsymbol{J}^{-1}(\hat{\Theta})$. The multivariate normal distribution with zero mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{I}^{-1}(\Theta)$ is used to construct confidence intervals for the $T-R\{Y\}$ – PS family parameters. The approximate $100(1-\alpha)\%$ two-sided confidence interval for the parameters θ and ξ are given by

$$\hat{\theta}\pm Z_{\alpha/2}\sqrt{\boldsymbol{I}_{\theta\theta}^{-1}(\hat{\Theta})}, \qquad \hat{\xi}\pm Z_{\alpha/2}\sqrt{\boldsymbol{I}_{\xi\xi}^{-1}(\hat{\Theta})},$$

respectively, where $\boldsymbol{I}_{\theta\theta}^{-1}(\hat{\Theta})$ and $\boldsymbol{I}_{\xi\xi}^{-1}(\hat{\Theta})$ are diagonal elements of $\boldsymbol{I}^{-1}(\hat{\Theta})$ and $Z_{\alpha/2}$ is the upper $(\alpha/2)^{th}$ percentile of a standard normal distribution.

A specific member from the new family: the Gumbel–Weibull {logistic}–Poisson (GUWELOP) distribution

Taking T, R, and Y as random variables following the Gumbel, Weibull and logistic distributions, respectively, Al-Aqtash et al. [29] defined the Gumbel–Weibull {logistics} (GW) distribution by the cdf and pdf expressed respectively as

$$F_{GW}(x) = \exp\left\{-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}}-1\right)^{-1/\gamma}\right\}, \tag{26}$$

$$f_{GW}(x) = \frac{\alpha\beta}{\lambda\gamma}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{\left(\frac{x}{\lambda}\right)^{\alpha}}\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}}-1\right)^{-1-1/\gamma}\exp\left\{-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}}-1\right)^{-1/\gamma}\right\}, \tag{27}$$

$$x>0, \alpha,\beta,\lambda,\gamma>0.$$

Taking the power series distribution as the Poisson distribution with properties as specified in Table 1 and substituting (26) and (27) into (4) and (5), we define the Gumbel – Weibull {logistic} Poisson (GUWELOP) distribution by the cdf and pdf given respectively by

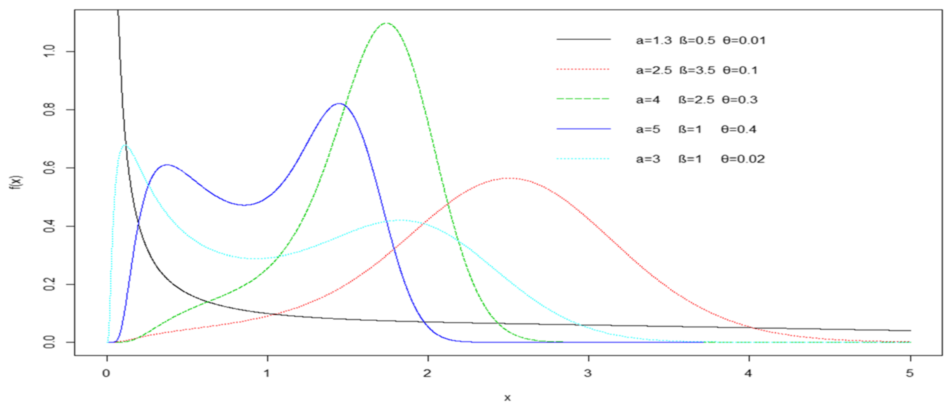

Fig. 1 GUWELOP density for varying parameter values ($\lambda = 1, \gamma = 6$)

$$F_{\mathrm{GUWELOP}}(x) = 1 - \frac{\exp\left\{\theta\left[1 - \exp\left(-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1\right)^{-\frac{1}{\gamma}}\right)\right]\right\} - 1}{e^{\theta} - 1}, \tag{28}$$

$$f_{\mathrm{GUWELOP}}(x) = \frac{\alpha\beta\theta}{\lambda\gamma(e^{\theta}-1)}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{\left(\frac{x}{\lambda}\right)^{\alpha}}\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1\right)^{-1-\frac{1}{\gamma}} \exp\left\{-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1\right)^{-\frac{1}{\gamma}}\right\} \times$$

$$\exp\left\{\theta\left[1 - \exp\left(-\beta\left(e^{\left(\frac{x}{\lambda}\right)^{\alpha}} - 1\right)^{-1/\gamma}\right)\right]\right\}, \tag{29}$$

$$x > 0, \alpha, \beta, \lambda, \gamma > 0, \theta \epsilon \mathbb{R}.$$

A graph of the pdf of the GUWELOP distribution is shown in Fig. 1. The graph of the pdf reveals that the GUWELOP density can be right-skewed, left-skewed, almost symmetric, and bimodal. To buttress the applicability of members of the new family in modeling complex real life data, the GUWELOP distribution is used to fit a multi-modal data set. The data set represents Kevlar 49/epoxy strands failure times data (pressure at 70%) reported in Al-Aqtash et al. [29] The data are multimodal, platykurtic, and approximately symmetric. (Skewness = 0.1, kurtosis = − 0.79). The data set is given in Table 8. The maximum likelihood method is used to fit the GUWELOP distribution, GW distribution, and the beta-normal (BN) distribution (Eugene et al. [11] to the data set. The results of the fit and other summary statistics are presented in Table 9. The graph of the fitted densities alongside the histogram of the data set is shown in Fig. 2.

Results from Table 9 show that the three distributions provided good fits to the data set since all the distributions have high p values of the K–S statistics. However, The GUWELOP distribution has the highest p value and hence provided the best fit to the data. This application suggests the adequacy of the GUWELOP distribution in fitting multi-modal data sets.

Table 8 Kevlar 49/epoxy strands failure times data (pressure at 70%)

1051, 1337, 1389, 1921, 1942, 2322, 3629, 4006, 4012, 4063, 4921, 5445, 5620, 5817, 5905, 5956, 6068, 6121, 6473, 7501, 7886, 8108, 8546, 8666, 8831, 9106, 9711, 9806, 10205, 10396, 10861, 11026, 11214, 11362, 11604, 11608, 11745, 11762, 11895, 12044, 13520, 13670, 14110, 14496, 15395, 16179, 17092, 17568, 17568

Table 9 Maximum likelihood estimates for Kevlar 49/epoxy strands failure times data (pressure at 70%)

Distribution	GW*	BN*	GUWELOP
Parameter estimates	$\hat{\alpha} = 2.6948$ (0.8101)	$\hat{a} = 0.1626$ (0.1039)	$\hat{\alpha} = 2.0433$ (0.8129)
	$\hat{\gamma} = 4.11091$ (1.0456)	$\hat{b} = 0.1157$ (0.0199)	$\hat{\gamma} = 3.6464$ (1.9910)
	$\hat{\beta} = 1.3118$ (0.5144)	$\hat{\mu} = 7826$ (1759.97)	$\hat{\beta} = 1.5924$ (1.2346)
	$\hat{\lambda} = 6165.69$ (1749.51)	$\hat{\sigma} = 1339.35$ (245.62)	$\hat{\lambda} = 4546.90$ (2220.66)
			$\hat{\theta} = -0.6017$ (2.5271)
Log likelihood	−478.51	−480.52	−478.8681
AIC	965.0	969.0	967.7362
K–S	0.0749	0.0797	0.0703
p value	0.9462	0.9144	0.9549

(Standard error of estimates in parenthesis)
*Maximum likelihood estimates, loglikelihood, AIC, *K–S* statistic, and its *p* value of the GW and BN distributions were obtained from Al-Aqtash et al. [29]

Summary and conclusion

A new family of probability distributions called the T–R $\{Y\}$—power series family of distributions has been introduced in this paper. The new family was realized by compounding the T–R $\{Y\}$ family of distribution and the power series family. Several mathematical properties of the new family were explored alongside the maximum likelihood method for the estimation of the parameters of the new family. A special member of the new family called the Gumbel–Weibull {logistics} Poisson distribution was defined and applied to a real data set in order to buttress the applicability of members of the new family in fitting real life data sets. Finally, we hope that the new family will attract usage in complex applications in the literature on compounded family of probability distributions.

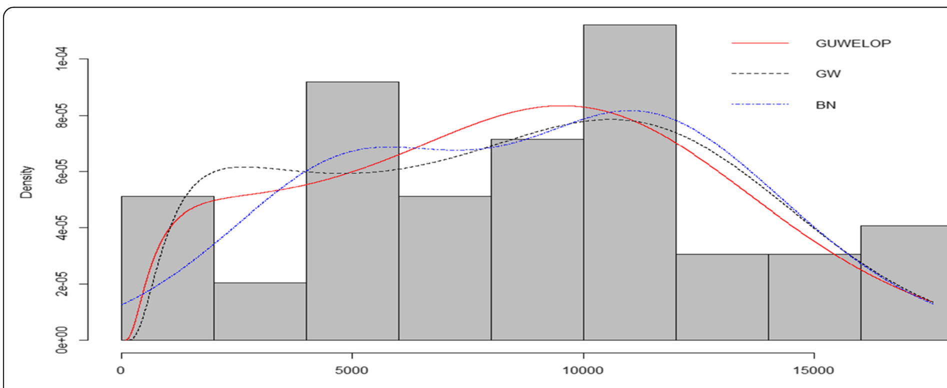

Fig. 2 Histogram and fitted densities of Kevlar 49/epoxy strands failure times data (pressure at 70%)

Abbreviations
AIC: Akaike Information Criterion; BN: beta normal; cdf: cumulative distributions function; GUWELOP: Gumbel – Weibull {logistic} Poisson; GW: Gumbel – Weibull; K – S: Kolmogorov –Smirnov; mgf: moment generating function; pdf: probability density function; T – R {Y} – B: T – R {Y} – binomial; T – R {Y} – G: T – R {Y} – geometric; T – R {Y} – L: T – R {Y} – logarithmic; T – R {Y} – P: T – R {Y} – Poisson

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