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Multiset concepts in two-universe approximation spaces

O. A. Embaby[1*] and Nadya A. Toumi[2]

* Correspondence: embabyosama@yahoo.com; embaby@science.tanta.edu.eg
[1]Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt
Full list of author information is available at the end of the article

Abstract

Rough set theory over two universes is a generalization of rough set model to find accurate approximations for uncertain concepts in information systems in which uncertainty arises from existence of interrelations between the three basic sets: objects, attributes, and decisions.
In this work, multisets are approximated in a crisp two-universe approximation space using binary ordinary relation and multi relation. The concept of two universe approximation is applied for defining lower and upper approximations of multisets. Properties of these approximations are investigated, and the deviations between them and corresponding notions are obtained; some counter examples are given. The suggested notions can help in the modification of the decision-making for events in which objects have repetitions such as patients visiting a doctor more than one time; an example for this case is given.

Keywords: Rough set, Multiset, Two universes approximation space

Mathematics Subject Classification: 54A05, 03E20, 68 U35

Introduction

A multiset is an unordered collection of objects in which, unlike the standard Cantorian set, the object is allowed to repeat. The word "multiset" often shortened to "mset" abbreviates the term "multiple membership set." In 1986, multiset theory was introduced by Yager [1]. Generalizations of the multiset concept were formalized by Blizard [2, 3]. Applications of multisets to rough approximations were studied by Miyamoto [4]. Over the years, besides the sporadic evidence of the applications of multisets in logic, linguistics, and physics, a great number of them are witnessed in mathematics and computer science. An overview of the applications of multisets is presented by Singh et al. [5]. Algebraic structures for the multiset space were constructed by Ibrahim et al. [6]. Girish and John introduced multiset topologies induced by multiset relations and the continuity between multiset topological spaces [7, 8]. El-Sheikh et al. introduced separation axioms on multiset topological spaces and operators on multiset bitopological spaces [9, 10]. The concepts of the exterior and boundary in the multiset topological space were introduced by Das and Mahanta [11]. Topological approximations of multisets are introduced by Abo-Tabl [12].

The rough set theory was proposed by Pawlak [13, 14] for the study of intelligent systems characterized by insufficient and incomplete information. The rough set theory has been applied in artificial intelligence, medical diagnosis, pattern recognition, data mining, conflict analysis, and algebra [15–23]. Wong, Wang, and Yao generalized the rough set model using two distinct but related universes [24]. The formulation and interpretation of U and V and the compatibility relation between the two universes depend very much on the available knowledge and the domain of applications. For example, in a medical diagnosis system, U can be a set of symptoms and V a set of diseases. Thus, uncertainty arises when describing the interrelations between symptoms and diseases in clinical settings. In a specific group of patients, each patient may show many symptoms, just as each disease could have many symptoms.

Shen et al. [25] researched the variable precision rough set model over two universes. Yan et al. [26] studied the model of rough set over dual universe. Fuzzy rough set models over two universes were studied by Weihua et al. [27]. Many advances of the rough set model over two universes can be found in literature [28–33]. In 2019, Sun et al. [34] provided the theoretical model of multi granulation vague rough set over two universes. Another is to try making a new way to handle group decision-making problems under uncertainty based on multi granulation vague rough set theory and methodologies over two universes.

Grish et al. [35–37] applied multisets for constructing approximations for rough multisets in information multi systems, rough multisets, and its multiset topology and rough multiset relations.

The rest of the paper is organized as follows: In the "Preliminaries" section, basic concepts used in the work are presented. The purpose of the "Approximation of multisets in crisp approximation space" section is to study approximations of rough multiset in two-universe approximation space. While the "Approximation based on multi binary relation" section contains an application for using multi binary relation for rough set approximation.

Preliminaries

This section is devoted to present the basic concepts and properties of rough sets and multisets.

Definition 2.1 [37] An mset drawn from the set A is represented by the count function C_M defined as $C_M : A \longrightarrow N$, where N is the set of all non-negative integers. Here, $C_M(a)$ is the number of occurrences of the element a in the mset M. The mset M is drawn from set $A = \{a_1, a_2, \ldots, a_n\}$ and is written as $M = \{m_1/a_1, m_2/a_2, \ldots,\ \ m_n/a_n\}$, where m_i is the number of occurrence of the element a_i, $i = 1, 2, \ldots, n$ in the mset M.

Definition 2.2 [37]. A domain A is defined as a set of elements from which msets are constructed. The mset space $[A]^\omega$ is the class of all msets drawn from the set A so that no element in the mset occurs more than ω times.

If $A = \{a_1, a_2, \ldots, a_n\}$, then $[A]^\omega = \{\{m_1/a_1, m_2/a_2, \ldots,\ \ m_n/a_n\} : m_i \in \{0, 1, 2, \ldots, \omega\}, i = 1, 2, \ldots, n\}$.

The mset space $[A]^\infty$ is the class of all msets over a domain A such that there is no limit in the number occurrences of an element in an mset.

Definition 2.3 [37] Let M and N be two msets drawn from a set A. Then:

1. $M = N$ if $C_M(a) = C_N(a) \quad \forall\, a \in A$
2. $M \subseteq N$ if $C_M(a) \le C_N(a) \quad \forall\, a \in A$
3. $P = M \cup N$ if $C_P(a) = \max\{C_M(a), C_N(a)\} \quad \forall\, a \in A$

4. $P = M \cap N$ if $C_P(a) = min\{C_M(a), C_N(a)\}$ $\forall a \in A$
5. $P = M \oplus N$ if $C_P(a) = min\{C_M(a) + C_N(a), \omega\}$ $\forall a \in A$
6. $P = M \ominus N$ if $C_P(a) = max\{C_M(a) - C_N(a), 0\}$ $\forall a \in A$, where $\oplus$ and $\ominus$ represent mset addition and subtraction, respectively.

Definition 2.4 [37] Let M be an mset drawn from a set A. The support set of M is a subset of A defined by $M^* = \{a \in A : C_M(a) > 0\}$, i.e., M^* is an ordinary set and is also called the root set of M.

Definition 2.5 [37] Let M be an mset drawn from a set A. If $C_M(a) = 0$ $\forall a \in A$, then M is called the empty mset and denoted by $\varnothing$.

Definition 2.6 [37] Let M be an mset drawn from a set A and $[A]^\omega$ be the mset space defined over A. Then, for any mset $M \in [A]^\omega$, the complement M^c of M in $[A]^\omega$ is an element of $[A]^\omega$ such that $C_{M^c}(a) = \omega - C_M(a)$ $\forall a \in A$.

Definition 2.7 [37] The cardinality of an mset M drawn from a set A is defined by$Card\ M = \sum_{a \in A} C_M(a)$. It is also denoted by $|M|$.

Notation 2.1 [7] Let $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ be an mset drawn from the set $X = \{x_1, x_2, \ldots, x_n\}$ with x appearing m times in M. It is denoted by $x \in^m M$. The entry of the form $(m/x, n/y)/k$ denotes that x is repeated m times, y is repeated n times, and the pair (x, y) is repeated k times. The counts of the members of the domain and co-domain vary in relation to the counts of the x coordinate and y coordinate in $(m/x, n/y)/k$. For this purpose, let the notation $C_1(x, y)$ denotes the count of the first co-ordinate in the ordered pair (x, y), and $C_2(x, y)$ denotes the count of the second co-ordinate in (x, y).

Definition 2.8 [7] Let M_1 and M_2 be two msets drawn from a set X; then, the Cartesian product of M_1 and M_2 is defined by $M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1 \ , \ y \in^n M_2\}$.

Definition 2.9 [7] A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has a count, the product of $C_1(x, y)$ and $C_2(x, y)$. We denote m/x related to n/y by $m/xRn/y$.

Definition 2.10 [38] Let (U, V, R) be a two-universe approximation space. Then, the set-valued mappings F and G represent the successor and predecessor neighborhood operators, respectively, defined as follows:

$F : U \longrightarrow P(V)$, $F(a) = \{b \in V : (a, b) \in R\}$, $G : V \longrightarrow P(U)$, $G(b) = \{a \in U : (a, b) \in R\}$.

F and G can be naturally extended to a mapping from $P(U)$ to $P(V)$ (resp. $P(V)$ to $P(U)$) which are also denoted by F and G:

$F : P(U) \longrightarrow P(V)$, $F(A) = \cup \{F(a) : a \in A\}$, $G : P(V) \longrightarrow P(U)$, $G(Y) = \cup \{G(b) : b \in A\}$.

Lemma 2.1 [38] Let (U, V, R) be a two-universe approximation space, if R is a strong inverse serial relation, then for all $a_1, a_2 \in U$, $F(A_1) \cap F(A_2) \neq \phi$ implies that $F(a_1) = F(a_2)$.

Proposition 2.1 [39] Let R be an arbitrary binary relation on U. Then, $\forall A \in P(U)$:

(i) R is reflexive $\Leftrightarrow \underline{R}_s(A) \subseteq A \Leftrightarrow A \subseteq \bar{R}_s(A)$

(ii) R is symmetric $\Leftrightarrow A \subseteq \underline{R}_s(\bar{R}_s(A)) \Leftrightarrow \bar{R}_s(\underline{R}_s(A)) \subseteq A$

(iii) R is transitive $\Leftrightarrow \underline{R}_s(A) \subseteq \underline{R}_s(\underline{R}_s(A)) \Leftrightarrow \bar{R}_s(\bar{R}_s(A)) \subseteq \bar{R}_s(A)$

(iv) R is Euclidean $\Leftrightarrow \bar{R}_s(A) \subseteq \underline{R}_s(\bar{R}_s(A)) \Leftrightarrow \bar{R}_s(\underline{R}_s(A)) \subseteq \underline{R}_s(A)$

Approximation of multisets in crisp approximation space

Definition 3.1 Let U and V be two finite non-empty universes of discourse and $R \in P(U \times V)$ be a binary relation from U to V. The ordered triple (U, V, R) is called a (two-universe) approximation space. Let $B \in [V]^w$ be a multi set drawn from V.

The lower and upper approximation of B, $\underline{R}_s(B)$ and $\bar{R}_s(B)$, with respect to the approximation space are multi set of U whose membership functions, for each $a \in U$, are defined, respectively, by:

$$C_{\underline{R}_s(B)}(a) = \min\{C_B(b) : b \in F(a)\}$$

$$C_{\bar{R}_s(B)}(a) = \max\{C_B(b) : b \in F(a)\}$$

where $F(a)$ is the successor neighborhood of a.

The ordered set pair $(\underline{R}_s(B), \bar{R}_s(B))$ is referred to as a generalized rough multiset with respect to successor neighborhood, and $\underline{R}_s : P(V) \longrightarrow P(U)$ and $\bar{R}_s : P(V) \longrightarrow P(U)$ are referred to as lower and upper generalized rough multi approximation operators, respectively.

Definition 3.2 Let (U, V, R) be a two-universe approximation space. Then, the lower and upper approximations of $A \in [U]^w$ are defined, respectively, as follows:

$$C_{\underline{R}_P(A)}(b) = \min\{C_A(a) : a \in G(b)\}$$

$$C_{\bar{R}_P(A)}(b) = \max\{C_A(a) : a \in G(b)\}$$

where $G(b)$ is the predecessor neighborhood of b.

The pair $(\underline{R}_P(A), \bar{R}_P(A))$ is referred to as a generalized rough multiset with respect to the predecessor neighborhood, and $\underline{R}_P : P(U) \longrightarrow P(V)$ and $\bar{R}_P : P(U) \longrightarrow P(V)$ are referred to as lower and upper rough multi approximation operators, respectively. If $\underline{R}_P(A) = \bar{R}_P(A)$, then A is called an exact multiset; otherwise, A is a rough multiset.

Proposition 3.1 In a two-universe model (U, V, R) with the binary relation R, the approximation operators $\underline{R}_P$ and $\bar{R}_P$ satisfy the following properties for all A, A_1, $A_2 \in [U]^w$:

(L_1) $\underline{R}_P(A) = (\bar{R}_P(A^c))^c$. (L_2) $\underline{R}_P(U) = V$.
(L_3) $\underline{R}_P(A_1 \cap A_2) = \underline{R}_P(A_1) \cap \underline{R}_P(A_2)$. (L_4) $\underline{R}_P(A_1 \cup A_2) \supseteq \underline{R}_P(A_1) \cup \underline{R}_P(A_2)$.
(L_5) $A_1 \subseteq A_2 \Longrightarrow \underline{R}_P(A_1) \subseteq \underline{R}_P(A_2)$. (U_1) $\bar{R}_P(A) = (\underline{R}_P(A^c))^c$. .
(U_2) $\bar{R}_P(\phi) = \phi$. (U_3) $\bar{R}_P(A_1 \cup A_2) = \bar{R}_P(A_1) \cup \bar{R}_P(A_2)$.
(U_4) $\bar{R}_P(A_1 \cap A_2) \subseteq \bar{R}_P(A_1) \cap \bar{R}_P(A_2)$. (U_5) $A_1 \subseteq A_2 \Longrightarrow \bar{R}_P(A_1) \subseteq \bar{R}_P(A_2)$.

Proof By the duality of approximation operators, we only need to prove the properties $L_1 - L_5$.

(L_1) For all $b \in V$, according to Definition 3.2, we can obtain:

$$\begin{aligned} C_{\left[\bar{R}_P(A^c)\right]^c}(b) &= w - \{\max\{C_{A^c}(a) : a \in G(b)\}\} = w - \{\max\{w - C_A(a) : a \in G(b)\}\} \\ &= w - \{w - \min\{C_A(\mathrm{a}) : a \in G(b)\}\} = w - w + \min\{C_A(a) : a \in G(b)\} \\ &= \min\{C_A(a) : a \in G(b)\} = C_{\underline{R}_P(A)}(b). \end{aligned}$$

Therefore, $\underline{R}_P(A) = (\bar{R}_P(A^c))^c$.

(L_2) Since $C_U(a) = 1 \; \forall \, a \in U$ and $G(b) \subseteq U$, the $\min\{C_U(a) : a \in G(b)\} = 1$. Thus, $C_{\underline{R}_P(U)}(b) = \min\{C_U(a) : a \in G(b)\} = 1$ for all $b \in V$. Therefore, $\underline{R}_P(U) = V$.

(L_3) Since $\forall b \in V$,

$$C_{\underline{R}_P(A_1\cap A_2)}(b) = min\{C_{(A_1\cap A_2)}(a) : a\in G(b)\} = min\{ min\{C_{A_1}(a), C_{A_2}(a)\} : a\in G(b)\}$$
$$= min\{ min\{C_{A_1}(a) : a\in G(b)\}, min\{C_{A_2}(a) : a\in G(b)\}\}$$
$$= min\left\{C_{\underline{R}_P(A_1)}(b), C_{\underline{R}_P(A_2)}(b)\right\} = C_{\underline{R}_P(A_1)\cap\underline{R}_P(A_2)}(b).$$

Therefore, $\underline{R}_P(A_1\cap A_2) = \underline{R}_P(A_1)\cap\underline{R}_P(A_2)$.

(L_4) For all $b\in V$, we can have:

$$C_{\underline{R}_P(A_1\cup A_2)}(b) = min\{C_{(A_1\cup A_2)}(a) : a\in G(b)\} = min\{ max\{C_{A_1}(a), C_{A_2}(a)\} : a\in G(b)\}$$
$$\geq max\{ min\{C_{A_1}(a) : a\in G(b)\}, min\{C_{A_2}(a) : a\in G(b)\}\}$$
$$= max\left\{C_{\underline{R}_P(A_1)}(b), C_{\underline{R}_P(A_2)}(b)\right\} = C_{\underline{R}_P(A_1)\cap\underline{R}_P(A_2)}(b).$$

Hence, $\underline{R}_P(A_1\cup A_2)\supseteq\underline{R}_P(A_1)\cup\underline{R}_P(A_2)$.

(L_5) Since $A_1\subseteq A_2$, then $\forall a\in U, C_{A_1}(a)\leq C_{A_2}(a)$. Thus, $C_{\underline{R}_P(A_1)}(b) = min\{C_{A_1}(a) : a\in G(b)\}\leq min\{C_{A_2}(a) : a\in G(b)\} = C_{\underline{R}_P(A_2)}(b)$.

Therefore, $\underline{R}_P(A_1)\subseteq\underline{R}_P(A_2)$.

The next proposition gives us characterizations of the rough multi lower and rough multi upper approximation operators based on different types of relations.

Proposition 3.2. Let $R\in P(U\times V)$ be an arbitrary binary relation. Then, $\forall A\in[U]^w$:

(i) R is inverse serial $\Leftrightarrow (L_6)\underline{R}_P(\phi) = \phi \Leftrightarrow (U_6)\bar{R}_P(U) = V \Leftrightarrow (LU)\underline{R}_P(A)\subseteq\bar{R}_P(A)$.

If $U = V$, then:

(ii) R is reflexive $\Leftrightarrow (L_7)\underline{R}_P(A)\subseteq A \Leftrightarrow (U_7)\ A\subseteq\bar{R}_P(A)$

(iii) R is symmetric $\Leftrightarrow (L_8)\ A\subseteq\underline{R}_P(\bar{R}_P(A)) \Leftrightarrow (U_8)\bar{R}_P(\underline{R}_P(A))\subseteq A$

(iv) R is transitive $\Leftrightarrow (L_9)\underline{R}_P(A)\subseteq\underline{R}_P(\underline{R}_P(A)) \Leftrightarrow (U_9)\bar{R}_P(\bar{R}_P(A))\subseteq\bar{R}_P(A)$

(v) R is left Euclidean $\Leftrightarrow (L_{10})\bar{R}_P(A)\subseteq\underline{R}_P(\bar{R}_P(A)) \Leftrightarrow (U_{10})\bar{R}_P(\underline{R}_P(A))\subseteq\underline{R}_P(A)$

Proof (i) Supposing that R is an inverse serial relation, then for any $b\in V$, we have $G(b)\neq\phi$. Thus, $C_{\underline{R}_P(\phi)}(b) = min\{C_\phi(a) : a\in G(b)\} = 0\forall b\in V$. Therefore, $\underline{R}_P(\phi) = \phi$.

Conversely, assuming that $\underline{R}_P(\phi) = \phi$,i.e., $C_{\underline{R}_P(\phi)}(b) = min\{C_\phi(a) : a\in G(b)\} = 0\ \forall b \in V$. If there exists $b_\circ\in V$ such that $G(b_\circ) = \phi$ then $C_{\underline{R}_P(\phi)}(b_\circ) = min\{C_\phi(a) : a\in G(b_\circ)\} = \min\{\quad\} = undefined$ which contradicts the assumption. Thus, $G(b)\neq\phi\,\forall\, b\in V$,i.e., R is an inverse serial. We can prove that R is an inverse serial if and only if $(U_6)\ \bar{R}_P(U) = V$ by the duality of approximation operators. For the third part, R is inverse serial *if and only if* $(LU)\ \underline{R}_P(A)\subseteq\bar{R}_P(A)$, and the proof is obvious.

(ii) By the duality, it is only to prove that R is reflexive if and only if $(L_7)\ \underline{R}_P(A)\subseteq A$. Since R is reflexive, then $\forall b\in V$, $b\in G(b)$, i.e., $min\{C_A(a) : a\in G(b)\}\leq C_A(b)$ which implies that $\underline{R}_P(A)\subseteq A$.

Conversely, assuming $\underline{R}_P(A)\subseteq A$ for all multi subset A of U. Because a crisp set is a special case of a multiset, then $\underline{R}_P(A)\subseteq A$ for all $A\subseteq U$ and by proposition 2.1, R is a reflexive relation.

(iii) Assuming that R is symmetric, then for all $a\in G(b)$, we have $b\in G(a)$. So, $max\{min\{C_A(c) : c\in G(a)\} : a\in G(b)\}\leq C_A(b)$.

Therefore, $\underline{R}_P(\bar{R}_P(A))\subseteq A$.

Conversely, assuming $\bar{R}_P(\underline{R}_P(A))\subseteq A$ for all multi subset A of U. Because a crisp set is a special case of a multiset, then $\bar{R}_P(\underline{R}_P(A))\subseteq A$ for all $A \subseteq U$ and by proposition 2.1, R is a symmetric relation. For the other statement, the proof is similar.

(iv) Supposing that R is a transitive relation, then for all $a \in G(b)$, we have $G(a) \subseteq G(b)$. Thus, $C_{\underline{R}_P(\underline{R}_P(A))}(b) = min\{ min\{C_A(a) : c\in G(a)\} : a\in G(b)\} \geq min\{ min\{C_A(c) : c\in G(b)\} : a\in G(b)\} = min\{C_A(c) : c\in G(b)\} = \underline{R}_{\mathcal{P}}(A)(b)$.

Therefore, $\underline{R}_P(A)\subseteq\underline{R}_P(\underline{R}_P(A))$.

The proof of the other side is similar to (iii).

(v) Assuming that R is a left Euclidean relation, then for all $a \in G(b)$, we have $G(b) \subseteq G(a)$. So, $C_{\bar{R}_P(\underline{R}_P(A))}(b) = max\{ min\{C_A(c) : c\in G(a)\} : a\in G(b)\} \leq max\{ min\{C_A(c) : c\in G(b)\} : a\in G(b)\} = min\{C_A(c) : c\in G(b)\} = C_{\underline{R}_P(A)}(b)$.

Therefore, $\bar{R}_P(\underline{R}_P(A))\subseteq\underline{R}_P(A)$.

The proof of the other side is like (iii).

Remark 3.1 If $R \in P(U \times V)$ is a serial relation in a two-universe approximation space (U, V, R), then the properties L_6, U_6, and LU are not true in general, as shown in the following example:

Example 3.1 Let $U = \{a_1, a_2, a_3, a_4\}$, $V = \{b_1, b_2, b_3, b_4, b_5\}$, and R be a binary relation from U to V defined as:

$$\mathrm{R} = \{(a_1, b_2), (a_1, b_4), (a_2, b_3), (a_2, b_4), (a_3, b_3), (a_4, b_1), (a_4, b_2)\}.$$

If $A \in [U]^w$ is a multiset drawn from U. Let $A = \{2/a_1, 3/a_2, 4/a_4\}$.

Then, we have:

	b_1	b_2	b_3	b_4	b_5
$C_{\underline{R}_{\mathcal{P}}(A)}(b)$	4	2	0	2	undefined
$C_{\bar{R}_{\mathcal{P}}(A)}(b)$	4	4	3	3	undefined
$C_{\underline{R}_{\mathcal{P}}(\varphi)}(b)$	0	0	0	0	undefined
$C_{\bar{R}_{\mathcal{P}}(U)}(b)$	**1**	1	1	1	undefined

Hence, $\underline{R}_P(\phi)\neq\phi$, $\bar{R}_P(U)\neq V$, and $\underline{R}_P(A)\neq\bar{R}_P(A)$, i.e., L_6, U_6, and LU do not hold.

Remark 3.2 Let R be any reflexive relation, then $\forall A \in [U]^w$ the properties $L_8 - L_{10}$ and $U_8 - U_{10}$ are not true in general. The following example shows this remark.

Example 3. 2 Let $U = \{a_1, a_2, a_3, a_4, a_5\}$ and R be a reflexive relation on U defined as $R = \{(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_2), (a_2, a_4), (a_3, a_3), (a_3, a_5),$

$$(a_4, a_2), (a_4, a_4), (a_5, a_2), (a_5, a_5)\}.$$

If A and B are multisets drawn from U defined as $A = \{2/a_2, 3/a_3, 4/a_5\}$ and $B = \{2/a_1, 3/a_2, 1/a_4, 4/a_5\}$, then we have:

	a_1	a_2	a_3	a_4	a_5
$C_{\underline{R}_P(B)}(a)$	2	1	0	1	0
$C_{\underline{R}_P(\underline{R}_P(A))}(a)$	1	0	0	1	0
$C_{\bar{R}_P(\underline{R}_P(A))}(a)$	0	3	3	0	3
$C_{\bar{R}_P(A)}(a)$	2	4	3	2	4

Approximation of multisets in crisp approximation space *(Continued)*

	a_1	a_2	a_3	a_4	a_5
$C_{\bar{R}_P(\bar{R}_P(B))}(a)$	4	4	0	4	4
$C_{\bar{R}_P(B)}(a)$	3	4	0	3	4
$C_{\underline{R}_P(\bar{R}_P(A))}(a)$	2	2	3	2	3

Hence, $A \not\subseteq \underline{R}_P(\bar{R}_P(A))$, $\underline{R}_P(A) \not\subseteq \underline{R}_P(\underline{R}_P(A))$, $\bar{R}_P(A) \not\subseteq \underline{R}_P(\bar{R}_P(A))$, $\bar{R}_P(\underline{R}_P(A)) \not\subseteq A$, $\bar{R}_P(\bar{R}_P(A)) \not\subseteq \bar{R}_P(A)$, $\bar{R}_P(\underline{R}_P(A)) \not\subseteq \underline{R}_P(A)$,i.e., $L_8 - L_{10}$, $U_8 - U_{10}$ do not hold.

Remark 3.3 Let R be any symmetric relation, then $\forall A \in [U]^w$ the properties L_6, L_7, L_9, L_{10}, U_6, U_7, U_9, U_{10} and LU are not true in general. The following example shows this remark.

Example 3.3 Let $U = \{a_1, a_2, a_3, a_4, a_5\}$ and R be a symmetric relation on U defined as $R = \{(a_1, a_1), (a_1, a_2), (a_2, a_1), (a_2, a_4), (a_4, a_2), (a_4, a_4), (a_5, a_5)\}$.

If A is a multiset drawn from U defined as $A = \{4/a_1, 2/a_2, 3/a_4, 1/a_5\}$, then we have:

	a_1	a_2	a_3	a_4	a_5
$\underline{R}_P(A)(a)$	2	3	undefined	2	1
$\underline{R}_P(\underline{R}_P(A))(a)$	2	2	undefined	2	1
$\bar{R}_P(\underline{R}_P(A))(a)$	3	**2**	undefined	3	1
$\bar{R}_P(A)(a)$	4	4	undefined	3	1
$\bar{R}_P(\bar{R}_P(A))(a)$	4	4	undefined	4	1
$\underline{R}_P(\bar{R}_P(A))(a)$	4	3	undefined	3	1
$\underline{R}_P(\varphi)(a)$	0	0	undefined	0	0
$\bar{R}_P(U)(a)$	1	1	undefined	1	1

Hence, $\underline{R}_P(\phi) \neq \phi$, $\underline{R}_P(A) \not\subseteq A$, $\underline{R}_P(A) \not\subseteq \underline{R}_P(\underline{R}_P(A))$, $\bar{R}_P(A) \not\subseteq \underline{R}_P(\bar{R}_P(A))$, $\bar{R}_P(U) \neq U$, $A \not\subseteq \bar{R}_P(A)$, $\bar{R}_P(\bar{R}_P(A)) \not\subseteq \bar{R}_P(A)$, $\bar{R}_P(\underline{R}_P(A)) \not\subseteq \underline{R}_P(A)$ and $\underline{R}_P(A) \not\subseteq \bar{R}_P(A)$, i.e., L_6, L_7, L_9, L_{10} and U_6, U_7, U_9, U_{10} and LU do not hold.

Remark 3.4 Let R be any transitive relation, then $\forall A \in [U]^w$ the properties L_6, L_7, L_8, L_{10}, U_6, U_7, U_8, U_{10} and LU do not hold in general. The following example shows this remark.

Example 3.4 Let $U = \{a_1, a_2, a_3, a_4, a_5\}$ and R be a transitive relation on U defined as $R = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_4, a_4), (a_5, a_2), (a_5, a_3)\}$.

If A is a multiset drawn from U defined as $A = \{3/\mathrm{a}_1, 4/\mathrm{a}_3, 2/\mathrm{a}_5\}$ and $B = \{3/a_1, 1/a_2, 2/a_4, 4/a_5\}$, then we have:

	a_1	a_2	a_3	a_4	a_5
$\underline{R}_P(A)(a)$	undefined	2	0	0	undefined
$\bar{R}_P(\underline{R}_P(B))(a)$	undefined	0	3	2	undefined
$\bar{R}_P(A)(a)$	undefined	3	3	0	undefined
$\underline{R}_P(B)(a)$	undefined	3	1	2	undefined
$\underline{R}_P(\bar{R}_P(A))(a)$	undefined	0	0	0	undefined
$\underline{R}_P(\varphi)(a)$	undefined	0	0	0	undefined
$\bar{R}_P(U)(a)$	undefined	1	1	1	undefined

Hence, $\underline{R}_P(\phi)\neq\phi$, $\underline{R}_P(A)\not\subseteq A$, $A\not\subseteq\underline{R}_P(\bar{R}_P(A))$, $\bar{R}_P(A)\not\subseteq\underline{R}_P(\bar{R}_P(A))$, $\bar{R}_P(U)\neq V$, $A\not\subseteq\bar{R}_P(A)$, $\bar{R}_P(\underline{R}_P(A))\not\subseteq A$, $\bar{R}_P(\underline{R}_P(A))\not\subseteq\underline{R}_P(A)$ and $\underline{R}_P(A)\not\subseteq\bar{R}_P(A)$, i.e., L_6, L_7, L_8, L_{10}, U_6, U_7, U_8, U_{10} and LU do not hold.

Definition 3.4 A multi constant $\hat{\alpha}$ is a multiset in U defined as:

$$C_{\hat{\alpha}}(a)=\alpha\forall a\in U,\alpha\in N.$$

Proposition 3.3 Let $(U,\ V,R)$ be a two- universe approximation space, the rough multi lower and upper approximation operators have the following properties for all $A_j\in[U]^w$, $j\in J$ which is an finite index set and for all $\alpha\in\{1,2,3,\ldots\}$,

(i)$\underline{R}_P\left(\cap_{j\in J}A_j\right)=\cap_{j\in J}\underline{R}_P\left(A_j\right)$.

(ii)$\underline{R}_P\left(\cup_{j\in J}A_{\mathrm{j}}\right)\supseteq\cup_{j\in J}\underline{R}_P\left(A_j\right)$.

(iii)$\underline{R}_P(A\cup\hat{\alpha})=\underline{R}_P(A)\cup\hat{\alpha}$.

(iv)$\bar{R}_P\left(\cup_{j\in J}A_j\right)=\cup_{j\in J}\bar{R}_P\left(A_j\right)$.

(v)$\bar{\mathrm{R}}_P\left(\cap_{\mathrm{j}\in\mathrm{J}}\mathrm{A}_{\mathrm{j}}\right)\subseteq\cap_{j\in J}\bar{R}_P\left(A_j\right)$.

Proof By the duality of approximation operators, we only need to prove the properties (i) – (iii).

(i) For each $b\in V$, we have:

$$C_{\underline{R}_P\left(\cap_{j\in J}A_j\right)}(b)=\ min\left\{C_{\left(\cap_{j\in J}A_j\right)}(a):(a)\in G(b)\right\}$$

$$=\ min\{\ min\{C_{A_j}(a):j\in J\}:a\in G(b)\}$$

$$=\ min\{\ min\{C_{A_j}(a):a\in G(b)\}:j\in J\}=\ min\left\{C_{\underline{R}_P\left(A_j\right)}(b):j\in J\right\}$$

$$=C_{\cap_{j\in J}\underline{R}_P\left(A_j\right)}(b).$$

(ii) Since $\forall(b)\in V$,

$$C_{\underline{R}_P\left(\cup_{j\in J}A_j\right)}(b)=\ min\left\{C_{\left(\cup_{j\in J}A_j\right)}(a):(a)\in G(b)\right\}$$

$$=\ min\{\ max\{C_{A_j}(a):j\in J\}:(a)\in G(b)\}$$

$$\geq\ min\{C_{B_j}(c):(c)\in G(b)\},\forall j\in J=C_{\underline{R}_P\left(A_j\right)}(b),\forall j\in J.$$

Therefore, $C_{\underline{R}_P(\cup_{j\in J}A_j)}(b)\geq\ max\{\underline{R}_P(A_j)(b),\forall j\in J\}=C_{\cup_{j\in J}\underline{R}_P(A_j)}(b)$.

(iii) For each $(b)\in V$, we have:

$$C_{\underline{R}_P(A\cup\hat{\alpha})}(b)=\ min\{C_{(A\cup\hat{\alpha})}(a):(a)\in G(b)\}=\ min\{\ max\{C_A(a),C_{\hat{\alpha}}(a)\}:(a)\in G(b)\}$$

$$=\ max\{\ min\{\mathrm{C}_A(a):a\in G(b)\},C_{\hat{\alpha}}(a)\}=C_{\left(\underline{R}_P(A)\cup\hat{\alpha}\right)}(b).$$

Proposition 3.4 Let (U,V,R) be a two-universe approximation space. Then, the following are equivalent $\forall\alpha\in N$

(i) R is an inverse serial relation,

(ii)$\underline{R}_P(\hat{\alpha})=\hat{\alpha}$, (iii)$\bar{R}_P(\hat{\alpha}) = \hat{\alpha}$.

Proof (i) $\Longrightarrow$ (ii) Let R be an inverse serial relation, then we have $\underline{R}_P(\hat{\alpha}) = \underline{R}_P(\hat{\alpha} \cup \phi)$ $= \hat{\alpha} \cup \underline{R}_P(\phi) = \hat{\alpha} \cup \phi = \widehat{\alpha}$.

(ii) $\Longrightarrow$ (iii) Coming from the duality of approximation operators.

(iii) $\Longrightarrow$ (i) Assuming $\bar{R}_P(\hat{\alpha}) = \hat{\alpha}$, since U is a special case of $\hat{\alpha}$ which is $\alpha = w$. Then by assumption, we have $\bar{R}_P(U) = V$, i.e., R is an inverse serial relation.

In the next three propositions, the connections of the approximation operators in definitions 2.7, and 3.1 are made, and the conditions under which these approximation operators made the equivalent are obtained.

Proposition 3.5 Let (U, V, R) be a two-universe approximation space, then the following holds for all $A \in [U]^w$and $B \in [V]^w$:

(i) $\bar{R}_s(\underline{R}_P(A)) \subseteq A, A \subseteq \underline{R}_s(\bar{R}_P(A))$, (iv) $\bar{R}_s(B) = \bar{R}_s(\underline{R}_P(\bar{R}_s(B)))$,

(ii) $\bar{R}_P(\underline{R}_s(B)) \subseteq B, B \subseteq \underline{R}_P(\bar{R}_s(B))$, (v) $\underline{R}_P(A) = \underline{R}_P(\bar{R}_s(\underline{R}_P(A)))$,

(iii) $\underline{R}_s(B) = \underline{R}_s(\bar{R}_P(\underline{R}_s(B)))$, (vi) $\bar{R}_P(A) = \bar{R}_P(\underline{R}_P(\bar{R}_P(A)))$

Proof (i) Since for every $a \in U$, we have either $F(a) = \phi$ or $F(a) \neq \phi$. If $F(a) = \phi$, then $C_{\bar{R}_s(\underline{R}_P(A))}(a) = \ max\{\ min\{C_A(a) : c \in G(b)\} : b \in F(a)\} = 0$ and hence $\bar{R}_s(\underline{R}_P(A)) \subseteq A$. If $A(a) \neq \phi$, then we have $a \in G(b)\ \forall\ b \in A(a)$. Thus, $max\{min\{C_A(c) : c \in G(b)\} b \in A(a)\} \leq$ $C_A(a)$, hence $\bar{R}_s(\underline{R}_P(A)) \subseteq A$. We can easily prove the other part by the duality of approximation operators.

(ii) is similar to (i).

(iii) – (vi) can be proved by the properties (i) and (ii).

Lemma 3.1 Let (U, V, R) be a two-universe approximation space, $b \in V$; if R is a strong inverse serial relation, then for all $a_1, a_2 \in G(b)$,

$$C_{\underline{R}_s(B)}(a_1) = C_{\underline{R}_s(B)}(a_2); C_{\bar{R}_s(B)}(a_1) = C_{\bar{R}_s(B)}(a_2).$$

Proof The proofs come directly from Lemma 2.1.

Proposition 3.6 Let (U, V, R) be a two-universe approximation space with a strong inverse serial relation, then the following holds for all $A \in [U]^w$ and $B \in [V]^w$:

(i) $\bar{R}_P(\underline{R}_P(B)) = \underline{R}_P(\bar{R}_P(B))$

(ii) $\underline{R}_P(\bar{R}_P(B)) = \bar{R}_P(\bar{R}_P(B))$.

Proof The proofs follow immediately from Lemma 3.1.

Proposition 3.7 Two pairs of lower approximation and upper approximation operators in definitions 2.7 and 3.2 are equivalent if and only if R is a symmetric relation.

Proof Let R be a symmetric relation on U, $A \in [U]^w$. Then for all $a \in U$, we have $F(a) = G(a)$, i.e.,

$$\begin{aligned} C_{\underline{R}_s(A)}(a) &= \ min\{C_A(b) : b \in F(a)\} \\ &= \ min\{C_A(b) : b \in G(a)\} = C_{\underline{R}_P(A)}(a). \end{aligned}$$

Conversely, assuming $\underline{R}_s(A) = \underline{R}_P(A)$, since by the proposition 3.4, we have $\bar{R}_P(\underline{R}_P(A)$ $) \subseteq A$, by proposition 3.1,and R is a symmetric relation.

Proposition 3.8 Let $G = (U, R)$ be a generalized approximation space and A be a multisubset of U. Then, the following holds:

(i) If R is symmetric then:

$$\underline{R}_P(A) = \underline{R}_P(\bar{R}_P(\underline{R}_P(A))); \bar{R}_P(A) = \bar{R}_P(\underline{R}_P(\bar{R}_P(A))).$$

(ii) If R is inverse serial and transitive then:

$$\underline{R}_P(A) \subseteq \underline{R}_P(\bar{R}_P(\underline{R}_P(A))); \bar{R}_P(A) \supseteq \bar{R}_P(\underline{R}_P(\bar{R}_P(A))).$$

Proof Obvious

Example 3.5 Let $U = \{a_1, a_2, a_3, a_4\}$ a set of four patients and $V = \{Fever(b_1)$, Headache(b_2), Stomachache (b_3), Cough(b_4), Myalgia $(b_5)\}$ be five symptoms,if $R = \{(a_1, b_2), (a_1, b_4), (a_2, b_3), (a_2, b_4), (a_3, b_3), (a_3, b_5), (a_4, b_1), (a_4, b_2), (a_4, b_5)\}$

is a relation relating patients to symptoms. Let $A = \{3/a_1, 0/a_2, 3/a_3, 5/a_4\}$ represents a multiset of patients and times of visiting the doctor. Thus, using definition 2.10, we have:

$$G(b_1) = \{a_4\}, G(b_2) = \{a_1, a_4\}, G(b_3) = \{a_2, a_3\}, G(b_4) = \{a_1, a_2\}, G(b_5) = \{a_3, a_4\}$$

and so, we get:

$$\underline{R}_P(A) = \left\{ {}^5\!/_{b_1}, {}^3\!/_{b_2}, {}^0\!/_{b_3}, {}^0\!/_{b_4}, {}^2\!/_{b_5} \right\} \textit{ and } \bar{R}_p(A) = \left\{ {}^5\!/_{b_1}, {}^5\!/_{b_2}, {}^2\!/_{b_3}, {}^3\!/_{b_4}, {}^5\!/_{b_5} \right\}.$$

If $A = \{a_1, a_3,\ \ a_4\}$. By using the class $U/R^{-1} = \{\{a_4\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_1, a_2\}, \{a_3, a_4\}\}$, the lower and upper approximations using rough sets on one universe U are $\underline{R}(A) = \{a_1, a_3,\ \ a_4\} = A$ and $\bar{R}(A) = \{a_1, a_2, a_3, a_4\} = U$. Clearly, this method does not have any deviations between the effectiveness of symptoms. But by using the multi approximations over the two universes U and V, we have degree of effectiveness of b_1 which is $\frac{5}{5}$, b_2 which is $\frac{3}{5}$, b_3 which is $\frac{0}{2}$, b_4 which is $\frac{0}{3}$, and b_5 which is $\frac{2}{5}$.

Approximation based on multi binary relation

In this section, we aim to approximate rough sets in multi approximation spaces, study their properties, and provide a counter example.

Definition 4.1 Let U and V be two finite non-empty universes of discourse. Let $M$and N be two multisets drawn from U and V, respectively. Let R be a multi binary relation from M to N. The ordered (U, V, M, N, R) is called a two-universe multi approximation space. For any crisp set $A \subseteq U$, the lower and upper approximations of A, $\underline{R}(A)$ and$\bar{R}(A)$, with respect to the multi approximation space, are multisets drawn from V whose count functions are defined respectively by:

For each $b \in V$,

$$C_{\underline{R}(A)}(b) = \ min\{m : (m/a) \in R(1/b), a \in A\}$$

$$C_{\bar{R}(A)}(b) = \ max\{m : (m/a) \in R(1/b), a \in A\}$$

If for all $b \in V,\ \ C_{\underline{R}(A)}(b) = C_{\bar{R}(A)}$, then the set A is definable (or exact) with respect to the multi approximation space (U, V, M, N, R)**.** Otherwise, the set A is rough with respect to the multi approximation space**.**

Proposition 4.1 In a multi approximation space $(U, V, , M, N, R)$, the multi approximation operators satisfy the following properties for all A, A_1, $A_2 \in P(U)$:

(L'_3) $\underline{R}(A_1{\cap}A_2){\subseteq}\underline{R}(A_1){\cap}\underline{R}(A_2)$ (U_3) $\bar{R}(A_1{\cup}A_2) = \bar{R}(A_1){\cup}\bar{R}(A_2)$

(L'_4) $\underline{R}(A_1{\cup}A_2) = \underline{R}(A_1){\cup}\underline{R}(A_2)(U_4)$ $\bar{R}(A_1{\cap}A_2){\subseteq}\bar{R}(A_1){\cap}\bar{R}(A_2)$

(L_5) $A_1{\subseteq}A_2{\Longrightarrow}\underline{R}(A_1){\subseteq}\underline{R}(A_2)$ (U_5) $A_1{\subseteq}A_2{\Longrightarrow}\bar{R}(A_1){\subseteq}\bar{R}(A_2)$.

(LU) $\underline{R}(A){\subseteq}\bar{R}(A)$.

Proof According to the duality of these properties, we only need to prove (L'_3), (L'_4), (L_5) and (LU).

(L_3) Since for all $b \in V$,

$$C_{\underline{R}(A_1\cap A_2)}(1/b) = \ min\{m : (m/a)R(1/b), a{\in}(A_1{\cap}A_2)\}$$

$$\leq \min\{\ min\{D : (m/a){\in}R(1/b), a{\in}A_1\},\ min\{\ m : (m/a){\in}R(1/b), a{\in}A_2\}\}$$

$$\leq \ min\{C_{\underline{R}(A_1)}(1/b), C_{\underline{R}(A_2)}(1/b)\}{\subseteq}\underline{R}(A_1){\cap}\underline{R}(A_2).$$

Hence, $\underline{R}_U(A_1{\cap}A_2){\subseteq}\underline{R}_U(A_1){\cap}\underline{R}_U(A_2)$.

(L_4) For all $b \in V$, we can have:

$$C_{\underline{R}(A_1\cup A_2)}(1/b) = \ min\{m : (m/a){\in}R(1/b), a{\in}(A_1{\cup}A_2)\}$$

$$= \ min\{\ max\{m : (m/a){\in}R(1/b), a{\in}(A_1), a{\in}(A_2)\}\}$$

$$= \ max\{\ min\{m : (m/a){\in}R(1/b), a{\in}A_1\},\ min\{m : (m/a){\in}(1/b), a{\in}A_2\}\}$$

$$= \ max\left\{C_{\underline{R}(A_1)}(1/b), C_{\underline{R}(A_2)}(1/b)\right\} = C_{\underline{R}(A_1)\cup\underline{R}(A_2)}(1/b).$$

Hence, $\underline{R}(A_1{\cup}A_2) = \underline{R}(A_1){\cup}\underline{R}(A_2)$.

(L_5) Since $A_1 \subseteq A_2$, then $\forall a{\in}U, A_1 C_{A_1}(a) \leq C_{A_2}(a)$. Thus, $C_{\underline{R}(A_1)}(1/b) = \ min\{m : (m/a){\in}R(1/b), a{\in}A_1\} \leq \ min\{m : (m/a){\in}R(1/b), a{\in}A_2\} = C_{\underline{R}(A_2)}(1/b)$.

Therefore, $\underline{R}(A_1){\subseteq}\underline{R}(A_2)$.

(LU) For all $b \in V$, we can have:

$$C_{\underline{R}(A)}(1/b) = \ min\{m : (m/a){\in}R(1/b), a{\in}A\}$$
$$\leq \ max\{m : (m/a){\in}R(1/b), a{\in}A\} = C_{\bar{R}(A)}(b).$$

Hence, $\underline{R}(A){\subseteq}\bar{R}(A)$.

Remark 4.1 If $R \in [M \times N]^w$ is a multi binary relation in a two-universe approximation space (U, V, M, N, R), then the following properties need not be true:

(L_1) $\underline{R}(A) = (\bar{R}(A^c))^c, (U_1)$ $\bar{R}(A) = (\underline{R}(A^c))^c$,

(L_2) $\underline{R}(U) = V, (U_2)$ $\bar{R}(\phi) = \phi$,

(L_3) $\underline{R}(A_1{\cap}A_2) = \underline{R}(A_1){\cap}\underline{R}(A_2), (U_6)$ $\bar{R}(U) = V$,

(L_4) $\underline{R}(A_1{\cup}A_2){\supseteq}\underline{R}(A_1){\cup}\underline{R}(A_2)$.

The following example shows this remark:

Example 4.1 Let $U = \{a_1, a_2, a_3, a_4,\ a_5,\ a_6,\ a_7\}$, $V = \{b_1, b_2, b_3, b_4, b_5\}$. Let M be a multiset drawn from U and N be a multiset drawn from V shath that $M = \{1/a_1, 2/a_2, 2/a_3, 1/a_4, 3/a_5, 2/a_6, 4/a_7\}$ and $N = \{2/b_1, , 3/b_3, 1/b_4, 4/b_5, 3/b_6\}$ and R be a multi binary relation from M to N defined as:

$$R = \{(1/a_1, 2/b_1)/2, (1/a_1, 3/b_3)/3, (1/a_1, 1/b_4)/1, (2/a_2, 3/b_3)/6, (2/a_2, 1/b_4)/2,$$
$$(2/a_2, 4/b_5)/8, (2/a_3, 2/b_1)/4, (2/a_3, 4/b_5)/8, (2/a_3, 3/b_6)/6, (1/a_4, 3/b_3)/3,$$
$$(1/a_4, 1/b_4)/1, (3/a_5, 2/b_1)/6, (3/a_5, 3/b_3)/9, (3/a_5, 1/b_4)/3, (3/a_5, 4/b_5)/12,$$
$$(2/a_6, 2/b_1)/4, (2/a_6, 3/b_3)/6, (2/a_6, 1/b_4)/2, (4/a_7, 2/b_1)/8, (4/a_7, 1/b_4)/4,$$
$$(4/a_7 4/b_5,)/16\}$$

If A is subset of U, defined as $A = A_1 = \{a_1, a_3, a_4, a_7\}$ and $A_2 = \{a_1, a_2, a_4, a_6\}$, then we have:

	b_1	b_2	b_3	b_4	b_5	b_6
$C_{\underline{R}(A_1)}(1/b)$	1	0	1	1	2	2
$C_{\overline{R}(A_1)}(1/b)$	4	0	1	4	4	2
$C_{\underline{R}(A_2)}(1/b)$	1	0	1	1	2	0
$C_{\overline{R}(A_2)}(1/b)$	2	0	2	2	2	0
$C_{\underline{R}(A_1 \cap A_2)}(1/b)$	1	0	1	1	0	0
$C_{\underline{R}(A_1)} \cap C_{\underline{R}(A_2)}(1/b)$	1	0	1	1	2	0
$C_{\underline{R}(A_1)} \cup C_{\underline{R}(A_2)}(1/b)$	1	0	1	1	2	2
$C_{\underline{R}(A_1 \cup A_2)}(1/b)$	1	0	1	1	2	2
$C_{(\underline{R}(A^c))^c}(1/b)$	5	6	5	5	4	4
$C_{(\overline{R}(A^c))^c}(1/b)$	3	6	3	3	3	6
$C_{\underline{R}(\varphi)}(1/b)$	0	undefined	0	0	0	0
$C_{\overline{R}(\varphi)}(1/b)$	0	undefined	0	0	0	0
$C_{\underline{R}(U)}(1/b)$	1	0	1	1	2	0
$C_{\overline{R}(U)}(1/b)$	4	0	3	4	4	2

Conclusion and future work

The multiset approximations suggested in this work can help to compute measures and ordering of effectiveness and certainty of concepts in information systems. More work on using multi relation to approximate rough multi sets will be discussed in the future. Also, the use of a relation between two universes (objects and attributes) can be extended to construct another relation between attributes and decisions for constructing a rough set model over three universes.

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Author details

[1]Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt. [2]Department of Mathematics, Faculty of Science, The 7th of April University, Al- Zawia, Libya.

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