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# The cordiality of the sum and union of two fourth power of paths and cycles

Ashraf Elrokh[1] and Aya Rabie[2*]

*Correspondence:
aya.ebrahim@inp.edu.eg
[2] Institute of National Planning, Cairo, Egypt
Full list of author information is available at the end of the article

**Abstract**

A simple graph is called cordial if it has 0-1 labeling that satisfies certain conditions. In this paper, we examine the necessary and sufficient conditions for cordial labeling of the sum and union of two fourth power of paths and cycles.

**Keywords:** Fourth power, Sum graph, Union graph, Cordial graph

**Mathematics Subject Classification:** 05C78, 05C75, 05C20

## Introduction

The field of graph theory plays an important role in various areas of pure and applied sciences. One of the main problems in this field is graph labeling which is an assignment of integers to the vertices or edges, or both, subject to certain conditions. It is a very powerful tool that eventually makes things in different fields very ease to be handled in mathematical way. While the labeling of graphs is perceived to be a primarily theoretical subject in the field of graph theory and discrete mathematics, it serves as models in a wide range of application like astronomy, coding theory, X-ray crystallography, circuit design and communication networks addressing [1]. An excellent reference for this purpose is the survey written by Gallian [2]. In this paper, all graphs are finite, simple and undirected. The original concept of cordial graphs is due to Cahit [3]. A mapping $f{:}V \to \{0,1\}$ is called *binary vertex labeling* of $G$ and $f(v)$ is called *the label of the vertex* $v$ *of* $G$ *under* $f$. For any edge $e = uv$, the induced edge labeling $f^{*}{:}E(G) \to \{0,1\}$ is given by $f^{*}(e) = |f(u) - f(v)|$, where $u, v \in V$. Let $v_f(i)$ be the numbers of vertices of $G$ labeled $i$ under $f$, and $e_f(i)$ be the numbers of edges of $G$ labeled $i$ under $f^{*}$ where $i \in \{0,1\}$. A binary vertex labeling of a graph $G$ is called *cordial* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *cordial* if it admits a cordial labeling. Cahit showed that each tree is cordial; a complete graph $K_n$ is cordial if and only if $n \leq 3$ and a complete bipartite graph $K_{n,m}$ is cordial for all positive integers $n$ and $m$ [3].

Let $G_1$ and $G_2$ are graphs. The sum of two graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, is defined as the graph with vertex set given by $V(G_1 + G_2) = V(G_1) \bigcup V(G_2)$ and its edge set is $E(G_1 + G_2) = E(G_1) \bigcup E(G_2) \bigcup J$, where $J$ consists of edges join each vertex of $G_1$ to every vertex of $G_2$. The union $G_1 \bigcup G_2$ of two graphs $G_1$ and $G_2$, is $G_1 \bigcup G_2 = (V(G_1) \bigcup V(G_2), E(G_1) \bigcup E(G_2))$. The fourth power of a graph $G$ is a graph with the same set of vertices as $G$, and an edge between two vertices iff there is a path of length at most 4 between them, such that $d(v_i, v_j) \leq 4$ and $i < j$. Diab [4, 5] has reported several results concerning the sum and union of the cycles $C_n$ and paths $P_m$ together with other specific graphs.

## Terminology and notations

A path with $m$ vertices and $m-1$ edges is denoted by $P_m$, and its fourth power $P_n^4$ has $n$ vertices and $4n-10$ edges. Also, a cycle with $n$ vertices and $n$ edges, denoted by $C_n$, and its fourth power $C_n^4$ has $n$ vertices and $4n-9$ edges. Let $L_{4r}$ denote the labeling $0011\,0011\ldots0011$ (repeated $r$-times). Let $L'_{4r}$ denote the labeling $0110\,0110\ldots0110$ (repeated $r$-times). The labeling $1100\,1100\ldots1100$ (repeated $r$-times) and labeling $1001\,1001\ldots1001$ (repeated $r$-times) are written as $S_{4r}$ and $S'_{4r}$, respectively. Let $M_r$ denote the labeling $0101\ldots01$, zero-one repeated $r$times if $r$ is even and $0101\ldots010$ if $r$ is odd; for example, $M_6 = 010101$ and $M_5 = 01010$. Let $M'_r$ denote the labeling $1010\ldots10$. We modify the labeling $M_r$ or $M'_r$ by adding symbols at one end or the other (or both). Also, $L_{4r}$ (or $L'_{4r}$ ) with extra labeling from right or left (or both sides).

If $L$ is a labeling for fourth power of paths $P_m$ and $M$ is a labeling for fourth power of paths $P_n$, then we use the notation $[L; M]$ for the labeling of the sum $P_m^4 + P_n^4$. Let $v_i$ and $e_i$ ( $i = 0, 1$) represent the numbers of vertices and edges, respectively, labeled by $i$. Let us denote $x_i$ and $a_i$ to be the numbers of vertices and edges labeled by $i$ for $P_m^4$. Also, let $y_i$ and $b_i$ be those for $P_n^4$. It is easy to verify that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. Also for $P_m^4 \cup P_n^4$, we use the same notation $[L; M]$ for the union $P_m^4 \cup P_n^4$, let $v_i$ and $e_i$ (for $i = 0, 1$) be the numbers of labels that are labeled by $i$ as before, also, $x_i$ and $a_i$ be the numbers of vertices and edges labeled by i for $P_m^4$, and let $y_i$ and $b_i$ be those for $P_n^4$. It is easy to verify that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$. To prove the result, we need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

## Results

### The cordiality of the sum of two fourth power of paths

In this subsection, we examine the cordiality of the sum of two fourth power of paths. To obtain this result, we use the following lemmas.

**Lemma 1** *If $n \equiv 0 (mod\ 4)$, then $P_n^4 + P_m^4$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r$, where $r \geq 2$. We consider the following cases.*

**Case 1.** $m \equiv 0 (mod\ 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 + P_{4s}^4$ by $[0L_{4r-4}011;\ 1_2L'_{4s-4}0_2]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r-5, y_0 = y_1 = 2s, b_0 = b_1 = 8s-5$ .

It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. As an example, Fig. 1 illustrates $P_8^4 + P_8^4$. Hence, $P_{4r}^4 + P_{4s}^4$ is cordial.

**Case 2.** $m \equiv 1 (mod\ 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 + P_{4s+1}^4$ by $[0L_{4r-4}011; 0_2L_{4s-4}101]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = 2s + 1, y_1 = 2s, b_0 = b_1 = 8s - 3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P_{4r}^4 + P_{4s+1}^4$ is cordial.

**Case 3.** $m \equiv 2 (mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 + P_{4s+2}^4$ by $[0L_{4r-4}011; 01_30S_{4s-4}0]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P_{4r}^4 + P_{4s+2}^4$ is cordial.

**Case 4.** $m \equiv 3 (mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P_{4r}^4 + P_{4s+3}^4$ by $[0L_{4r-4}011; 0_21L_{4s}]$. Therefore

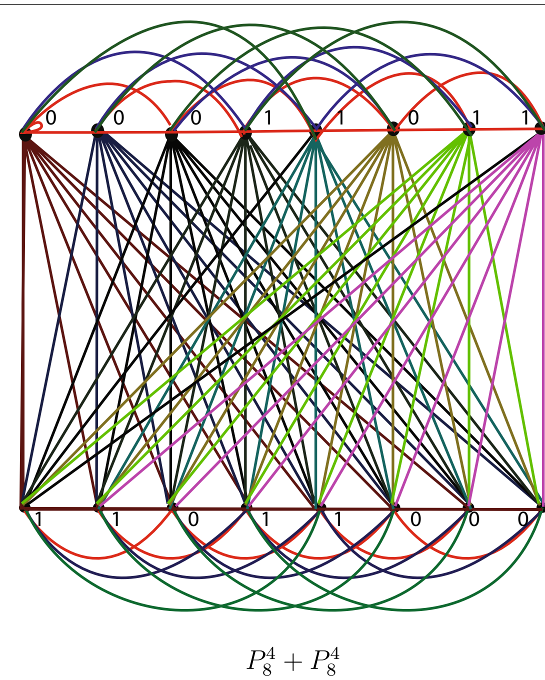


**Fig. 1** $P_8^4 + P_8^4$

$x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P^4_{4r} + P^4_{4s+3}$ is cordial. □

**Lemma 2** *If $n \equiv 1(mod\ 4)$, then $P^4_n + P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r + 1$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+1} + P^4_{4s+1}$ by $[0_2L_{4r-4}101; 1_2L'_{4s-4}010]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = 2s, y_1 = 2s + 1, b_0 = b_1 = 8s - 3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$. Hence, $P^4_{4r+1} + P^4_{4s+2}$ is cordial.

**Case 2**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+1} + P^4_{4s+2}$ by $[0_2L_{4r-4}101; 01_30S_{4s-4}0]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P^4_{4r+1} + P^4_{4s+2}$ is cordial.

**Case 3**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+1} + P^4_{4s+3}$ by $[0_2L_{4r-4}101; 1_2S_{4s}0]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = 2s + 1, y_1 = 2s + 2, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$. Hence, $P^4_{4r+1} + P^4_{4s+3}$ is cordial. □

**Lemma 3** *If $n \equiv 2(mod\ 4)$, then $P^4_n + P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r + 2$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+2} + P^4_{4s+2}$ by $[01_3 0S_{4r-4}0; 01_3 0S_{4s-4}0]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = a_1 = 8r - 1, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P^4_{4r+2} + P^4_{4s+2}$ is cordial.

**Case 2.** $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+2} + P^4_{4s+3}$ by $[01_3 0S_{4r-4}0; 0_2 1L_{4s}]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = a_1 = 8r - 1, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $P^4_{4r+2} + P^4_{4s+3}$ is cordial. □

**Lemma 4** *If $n, m \equiv 3(mod\ 4)$, then $P^4_n + P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r + 3$, where $r \geq 2$ and $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+3} + P^4_{4s+3}$ by $[0_2 1L_{4r}; 1_2 S_{4s}0]$. Therefore $x_0 = 2r+2, x_1 = 2r+1, a_0 = a_1 = 8r+1, y_0 = 2s+1, y_1 = 2s+2, b_0 = b_1 = 8s+1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$. Hence, $P^4_{4r+3} + P^4_{4s+3}$ is cordial.*

By considering all the lemmas mentioned in section "The cordiality of the sum of two fourth power of paths" we write the following theorem. □

**Theorem 1** *The sum of two fourth power of paths $P^4_n + P^4_m$ is cordial for all $n, m \geq 7$*

#### The cordiality of sum of two fourth power of cycles

In this subsection, we study the cordiality of sum of two fourth power of cycles.

**Lemma 5** *If $n \equiv 0(mod\ 4)$, then $C^4_n + C^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r$, where $r \geq 2$. We consider the following cases.*

**Case 1.** $m \equiv 0(mod\ 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $C^4_{4r} + C^4_{4s}$ by $[S'_{4r}; 1_3 M_{4s-6}0_3]$. Therefore

$x_0 = x_1 = 2r, a_0 = 8r-5, a_1 = 8r-4, y_0 = y_1 = 2s, b_0 = 8s-4, b_1 = 8s-5$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r} + C^4_{4s}$ is cordial.

**Case 2**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s+1$, where $s \geq 2$. Then we label the vertices of $C^4_{4r} + C^4_{4s+1}$ by $[1_3M_{4r-6}0_3; L_{4s}0]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-4, a_1 = 8r-5, y_0 = 2s+1, y_1 = 2s, b_0 = 8s-3, b_1 = 8s-2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r} + C^4_{4s+1}$ is cordial.

**Case 3**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s+2$, where $s \geq 2$. Then we label the vertices of $C^4_{4r} + C^4_{4s+2}$ by $[S'_{4r}; 0_3101_3M_{4s-6}]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-5, a_1 = 8r-4, y_0 = y_1 = 2s+1, b_0 = 8s, b_1 = 8s-1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r} + C^4_{4s+2}$ is cordial.

**Case 4**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s+3$, where $s \geq 1$. Then we label the vertices of $C^4_{4r} + C^4_{4s+3}$ by $[1_3M_{4r-6}0_3; L'_{4s}010]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-4, a_1 = 8r-5, y_0 = 2s+2, y_1 = 2s+1, b_0 = 8s+1, b_1 = 8s+2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r} + C^4_{4s+3}$ is cordial. □

**Lemma 6** *If $n \equiv 1(mod\ 4)$, then $C^4_n + C^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r+1$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s+1$, where $s \geq 2$. Then we label the vertices of $C^4_{4r+1} + C^4_{4s+1}$ by $[L_{4r}0; 1_3L'_{4s}0_2]$. Therefore $x_0 = 2r+1, x_1 = 2r, a_0 = 8r-3, a_1 = 8r-2, y_0 = 2s, y_1 = 2s+1, b_0 = 8s-2, b_1 = 8s-3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$. Hence, $C^4_{4r+1} + C^4_{4s+1}$ is cordial.

**Case 2**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C^4_{4r+1} + C^4_{4s+2}$ by $[L_{4r}0; 0_3 1 0 1_3 M_{4s-6}]$. Therefore $x_0 = 2r+1, x_1 = 2r, a_0 = 8r-3, a_1 = 8r-2, y_0 = y_1 = 2s+1, b_0 = 8s, b_1 = 8s-1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r+1} + C^4_{4s+2}$ is cordial.

**Case 3**. $m \equiv 3 (mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C^4_{4r+1} + C^4_{4s+3}$ by $[1_3 L'_{4r-2} 0; L'_{4s} 010]$. Therefore $x_0 = 2r, x_1 = 2r+1, a_0 = 8r-2, a_1 = 8r-3, y_0 = 2s+2, y_1 = 2s+1, b_0 = 8s+1, b_1 = 8s+2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = -1$. Hence, $C^4_{4r+1} + C^4_{4s+3}$ is cordial. □

**Lemma 7** *If $n \equiv 2(mod\ 4)$, then $C^4_n + C^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r + 2$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C^4_{4r+2} + C^4_{4s+2}$ by $[0_3 1_3 L'_{4r-4}; 0_3 1 0 1_3 M_{4s-6}]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = 8r - 1, a_1 = 8r, y_0 = y_1 = 2s + 1, b_0 = 8s, b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r+2} + C^4_{4s+2}$ is cordial.

**Case 2**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C^4_{4r+2} + C^4_{4s+3}$ by $[0_3 1 0 1_3 M_{4s-6}; L'_{4s} 010]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = 8r, a_1 = 8r - 1, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = 8s + 1, b_1 = 8s + 2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Hence, $C^4_{4r+2} + C^4_{4s+3}$ is cordial. □

By considering all the lemmas mentioned in section "The cordiality of sum of two fourth power of cycles" we write the following theorem.

**Theorem 2** *The sum of two fourth power of cycles $C^4_n + C^4_m$ is cordial for all $n, m \geq 7$ except at $(n, m) = (7, 7)$*

### The cordiality of union of two fourth power of paths

In this subsection, we examine the cordiality of the union of two fourth power of paths. To obtain this result, we use the following lemmas.

**Lemma 8** *If $n \equiv 0(mod\ 4)$, then $P_n^4 \cup P_m^4$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r$, where $r \geq 2$. We consider the following cases.*

**Case 1.** $m \equiv 0(mod\ 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 \cup P_{4s}^4$ by $[0L_{4r-4}011;\ 1_2L'_{4s-4}0_2]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = y_1 = 2s, b_0 = b_1 = 8s - 5$. As an example, Fig. 2 illustrates $p_8^4 \cup p_8^4$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P_{4r}^4 \cup P_{4s}^4$ is cordial.

**Case 2.** $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 \cup P_{4s+1}^4$ by $[0L_{4r-4}011;\ 0_2L_{4s-4}101]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = 2s + 1, y_1 = 2s, b_0 = b_1 = 8s - 3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P_{4r}^4 \cup P_{4s+1}^4$ is cordial.

**Case 3.** $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P_{4r}^4 \cup P_{4s+2}^4$ by $[0L_{4r-4}011;\ 01_30S_{4s-4}0]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that

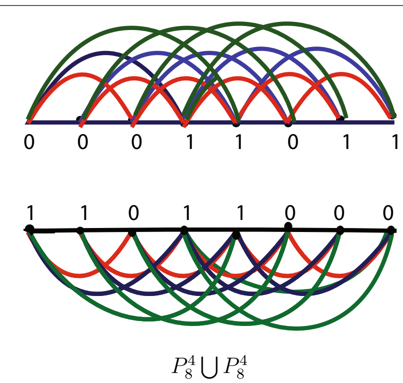


**Fig. 2** $P_8^4 \bigcup P_8^4$

$v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r} \cup P^4_{4s+2}$ is cordial.

**Case 4**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r} \cup P^4_{4s+3}$ by $[0L_{4r-4}011;\ 0_21L_{4s}]$. Therefore $x_0 = x_1 = 2r, a_0 = a_1 = 8r - 5, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r} \cup P^4_{4s+3}$ is cordial. □

**Lemma 9** *If $n \equiv 1(mod\ 4)$, then $P^4_n \cup P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r + 1$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s + 1$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+1} \cup P^4_{4s+1}$ by $[0_2L_{4r-4}101;\ 1_2L'_{4s-4}010]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = 2s, y_1 = 2s + 1, b_0 = b_1 = 8s - 3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+1} \cup P^4_{4s+1}$ is cordial.

**Case 2**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+1} \cup P^4_{4s+2}$ by $[0_2L_{4r-4}101;\ 01_30S_{4s-4}0]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+1} \cup P^4_{4s+2}$ is cordial.

**Case 3**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+1} \cup P^4_{4s+3}$ by $[0_2L_{4r-4}101;\ 1_2S_{4s}0]$. Therefore $x_0 = 2r + 1, x_1 = 2r, a_0 = a_1 = 8r - 3, y_0 = 2s + 1, y_1 = 2s + 2, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+1} \cup P^4_{4s+3}$ is cordial. □

**Lemma 10** *If $n \equiv 2(mod\ 4)$, then $P^4_n \cup P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r + 2$, where $r \geq 2$. We consider the following cases.*

**Case 1.** $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $P^4_{4r+2} \cup P^4_{4s+2}$ by $[01_30S_{4r-4}0; 01_30S_{4s-4}0]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = a_1 = 8r - 1, y_0 = y_1 = 2s + 1, b_0 = b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+2} \cup P^4_{4s+2}$ is cordial.

**Case 2.** $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+2} \cup P^4_{4s+3}$ by $[01_30S_{4r-4}0; 0_21L_{4s}]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = a_1 = 8r - 1, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = b_1 = 8s + 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+2} \cup P^4_{4s+3}$ is cordial. □

**Lemma 11** *If $n, m \equiv 3(mod\ 4)$, then $P^4_n \cup P^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r + 3$, where $r \geq 2$ and $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $P^4_{4r+3} \cup P^4_{4s+3}$ by $[0_21L_{4r}; 1_2S_{4s}0]$. Therefore $x_0 = 2r+2, x_1 = 2r+1, a_0 = a_1 = 8r+1, y_0 = 2s+1, y_1 = 2s+2, b_0 = b_1 = 8s+1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $P^4_{4r+3} \cup P^4_{4s+3}$ is cordial.*

By considering all the lemmas mentioned in section "The cordiality of union of two fourth power of paths" we write the following theorem. □

**Theorem 3** *The union of two fourth power of paths $P^4_n \cup P^4_m$ is cordial for all $n, m \geq 7$.*

### The cordiality of union of two fourth power of cycles

In this subsection, we examine the cordiality of the union of two fourth power of cycles. To obtain this result, we use the following lemmas.

**Lemma 12** *If $n \equiv 0(mod\ 4)$, then $C^4_n \cup C^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r$, where $r \geq 2$. We consider the following cases.*

**Case 1.** $m \equiv 0(mod\ 4)$.

Suppose that $m = 4s$, where $s \geq 2$. Then we label the vertices of $C^4_{4r} \cup C^4_{4s}$ by $[S'_{4r}; 1_3M_{4s-6}0_3]$. Therefore

$x_0 = x_1 = 2r, a_0 = 8r-5, a_1 = 8r-4, y_0 = y_1 = 2s, b_0 = 8s-4, b_1 = 8s-5$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C_{4r}^4 \cup C_{4s}^4$ is cordial.

**Case 2**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s+1$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+1}^4$ by $[1_3 M_{4r-6} 0_3; L_{4s} 0]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-4, a_1 = 8r-5, y_0 = 2s+1, y_1 = 2s, b_0 = 8s-3, b_1 = 8s-2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C_{4r}^4 \cup C_{4s+1}^4$ is cordial.

**Case 3**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s+2$, where $s \geq 2$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+2}^4$ by $[S'_{4r}; 0_3 1 0 1_3 M_{4s-6}]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-5, a_1 = 8r-4, y_0 = y_1 = 2s+1, b_0 = 8s, b_1 = 8s-1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C_{4r}^4 \cup C_{4s+2}^4$ is cordial.

**Case 4**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s+3$, where $s \geq 1$. Then we label the vertices of $C_{4r}^4 \cup C_{4s+3}^4$ by $[1_3 M_{4r-6} 0_3; L'_{4s} 010]$. Therefore $x_0 = x_1 = 2r, a_0 = 8r-4, a_1 = 8r-5, y_0 = 2s+2, y_1 = 2s+1, b_0 = 8s+1, b_1 = 8s+2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C_{4r}^4 \cup C_{4s+3}^4$ is cordial. □

**Lemma 13** *If $n \equiv 1(mod\ 4)$, then $C_n^4 \cup C_m^4$ is cordial for all $n, m \geq 7$.*

***Proof***

*Suppose that $n = 4r+1$, where $r \geq 2$. We consider the following cases.*

**Case 1**. $m \equiv 1(mod\ 4)$.

Suppose that $m = 4s+1$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 \cup C_{4s+1}^4$ by $[L_{4r} 0; 1_3 L'_{4s-4} 0_2]$. Therefore $x_0 = 2r+1, x_1 = 2r, a_0 = 8r-3, a_1 = 8r-2, y_0 = 2s, y_1 = 2s+1, b_0 = 8s-2, b_1 = 8s-3$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C_{4r+1}^4 \cup C_{4s+1}^4$ is cordial.

**Case 2**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s+2$, where $s \geq 2$. Then we label the vertices of $C_{4r+1}^4 \cup C_{4s+2}^4$ by $[L_{4r} 0; 0_3 1 0 1_3 M_{4s-6}]$. Therefore

$x_0 = 2r+1, x_1 = 2r, a_0 = 8r-3, a_1 = 8r-2, y_0 = y_1 = 2s+1, b_0 = 8s, b_1 = 8s-1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C^4_{4r+1} \cup C^4_{4s+2}$ is cordial.

**Case 3**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C^4_{4r+1} \cup C^4_{4s+3}$ by $[1_3 L'_{4r-2} 0_2; L'_{4s} 010]$. Therefore $x_0 = 2r, x_1 = 2r+1, a_0 = 8r-2, a_1 = 8r-3, y_0 = 2s+2, y_1 = 2s+1, b_0 = 8s+1, b_1 = 8s+2$.
It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C^4_{4r+1} \cup C^4_{4s+3}$ is cordial. □

**Lemma 14** *If $n \equiv 2(mod\ 4)$, then $C^4_n \cup C^4_m$ is cordial for all $n, m \geq 7$.*

***Proof***
*Suppose that $n = 4r + 2$, where $r \geq 2$. The following cases will be examined.*

**Case 1**. $m \equiv 2(mod\ 4)$.

Suppose that $m = 4s + 2$, where $s \geq 2$. Then we label the vertices of $C^4_{4r+2} \cup C^4_{4s+2}$ by $[0_3 1_3 L'_{4r-4}; 0_3 101_3 M_{4s-6}]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = 8r - 1, a_1 = 8r, y_0 = y_1 = 2s + 1, b_0 = 8s, b_1 = 8s - 1$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C^4_{4r+2} \cup C^4_{4s+2}$ is cordial.

**Case 2**. $m \equiv 3(mod\ 4)$.

Suppose that $m = 4s + 3$, where $s \geq 1$. Then we label the vertices of $C^4_{4r+2} \cup C^4_{4s+3}$ by $[0_3 101_3 M_{4s-6}; L'_{4s} 010]$. Therefore $x_0 = x_1 = 2r + 1, a_0 = 8r, a_1 = 8r - 1, y_0 = 2s + 2, y_1 = 2s + 1, b_0 = 8s + 1, b_1 = 8s + 2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. Hence, $C^4_{4r+2} \cup C^4_{4s+3}$ is cordial. □

By considering all the lemmas mentioned in section "The cordiality of union of two fourth power of cycles" we write the following theorem.

**Theorem 4** *The union of two fourth power of cycles $C^4_n \cup C^4_m$ is cordial for all $n, m \geq 7$ except at $(n, m) = (7, 7)$.*

## Conclusion

In this paper we test the cordiality of the sum and union of two fourth power of paths and cycles. We found that $P^4_n + P^4_m$ and $P^4_n \cup P^4_m$ is cordial for all $n, m \geq 7$ and also $C^4_n + C^4_m$ and $C^4_n \cup C^4_m$ is cordial for all $n$, $m$ except at $(n, m) = (7, 7)$

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**Authors' contributions**
AR wrote the title, abstract, stability, graph the figures and conclusion and fixed many language errors. AER wrote the introduction and references. AER wrote the mathematical analysis. AR wrote the bifurcation analysis. AER wrote the numerical analysis. All authors read and approved the final manuscript.

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**Author details**
[1] Department of Mathematics, Faculty of Science, Menoufia University, Shebin El Kom, Egypt. [2] Institute of National Planning, Cairo, Egypt.

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