ORIGINAL RESEARCH   Open Access

Nano $\wedge_\beta$-sets and nano $\wedge_\beta$-continuity

M. Hosny[1,2]

Correspondence: monahosny@edu.
asu.edu.eg
[1]Department of Mathematics,
College of Science for Girls, King
Khalid University, Abha, Saudi
Arabia
[2]Department of Mathematics,
Faculty of Education, Ain Shams
University, Cairo, Egypt

Abstract

The concept of nano near open sets was originally proposed by Thivagar and Richard (Int. J. Math. Stat. Inven 1:31-37). The main aspect of this paper is to introduce a new sort of nano near open sets namely, nano $\wedge_\beta$-sets. Fundamental properties of these sets are studied and compared to the previous one. It turns out that every nano β-open set is a nano $\wedge_\beta$-set. So, nano $\wedge_\beta$-sets are an extension of the previous nano near open sets, such as nano regular open, nano α-open, nano semi-open, nano pre-open, nano γ-open, and nano β-open sets. Meanwhile, it is shown that the concepts of nano $\wedge_\beta$-sets and nano $\delta\beta$-open sets are different and independent. Based on these new sets, nano $\wedge_\beta$-continuous functions are defined and some results involving their characterizations are derived. In addition, the concepts of nano $\vee_\beta$-closure and nano $\wedge_\beta$-interior are presented. Their properties are used to introduce and study the nano $\wedge_\beta$-continuous functions.

Keywords: Nano topology, Nano $\wedge_\beta$-sets, Nano $\wedge_\beta$-continuity

2010 AMS Classification 54A05, 54B05, 54C05, 54C10

Introduction

Thivagar and Richard [1] established and constructed nano topological spaces, nano open sets, nano closed sets, nano interior, and nano closure. The nano α-open sets, nano semi-open sets, nano pre-open sets, and nano regular-open sets are also studied in [1]. After the work of Thivagar and Richard [1] on nano near open sets, various mathematicians turned their attention to the generalizations of these sets. In this direction, the nano β-open sets and some of their properties were discussed in [2] and shown that this notion was a generalization of the other types of nano near sets [1]. More importantly, Nasef et al. [3] provided new properties of some weak forms of nano open sets and studied the relationships between them. In 2018, Hosny [4] proposed the notion of nano $\delta\beta$-open sets as a generalization of nano β-open sets. Consequently, it was an extension of all the previous weak forms of nano open sets in [1]. Nano continuous functions were defined in terms of nano open sets in [5]. On the other hand, nano α-continuous, nano semi-continuous, nano pre-continuous, and nano γ-continuous were investigated in [6, 7]. The concept of the nano β-continuity was studied by Nasef et al. [3]. It was shown that this concept was an extension of the previous concepts of nano near continuous functions [5–7]. More recently, the notion of nano $\delta\beta$-continuous functions was introduced in [4].

This notion was a generalization of nano β-continuous. Therefore, it was a generalization of the other types of nano near continuous functions [5–7].

The purpose of this paper is to continue the research along these directions. The intention of the present work is to generalize the notion of nano β-open sets by proposing a study of a new structure in nano topology which is the nano $\wedge_\beta$-sets. It should be noted that the generalization of nano β-open sets by using $\wedge_\beta$-sets is very different from the generalization of nano β-open sets by using nano $\delta\beta$-open sets [4]. The main difference states that the family of all nano $\delta\beta$-open sets does not form a topology, as the intersection of two nano $\delta\beta$-open sets need not be a nano $\delta\beta$-open set as shown in [4]. While, the family of all nano $\wedge_\beta$-sets forms a topology as it is shown in this work. This new notion is not only an extension of nano β-open sets, but also can be regarded as a generalization of the other kinds of nano near open sets. This paper is organized as follows: The "Preliminaries" section contains the basic concepts of nano topological spaces. The nano $\wedge_\beta$-sets and their properties are presented in the "Nano $\wedge_\beta$-sets" section. The basic nano topological properties of this concept are also studied in this section. The relationships between the nano $\wedge_\beta$-sets and nano β-open sets are revealed through Lemma 3.2. Moreover, it is shown that the concepts of nano $\wedge_\beta$-sets and nano $\delta\beta$-open sets are different and independent (see Examples 3.1 and 3.2). At the end of this section, we draw a diagram to describe the relationships among nano $\wedge_\beta$-sets and the previous nano near sets. The objective of the "Nano $\wedge_\beta$-sets and lower and upper approximations" section is to obtain various forms of nano $\wedge_\beta$-sets corresponding to different cases of approximations. In the "Nano $\wedge_\beta$-continuity" section, nano $\wedge_\beta$-continuous functions are introduced as a generalization of nano β-continuous functions. Additionally, Remark 5.2 shows that the difference between nano $\wedge_\beta$-continuous functions and nano $\delta\beta$-continuous functions. The notion and the fundamental properties of nano $\vee_\beta$-closure and nano $\wedge_\beta$-interior are given. The nano $\wedge_\beta$-continuous functions are redefined by using the concepts of nano closed sets, nano closure, nano interior, nano $\vee_\beta$-sets (the complement of nano $\wedge_\beta$-sets), nano $\vee_\beta$-closure and nano $\wedge_\beta$-interior. The "Conclusions" section concludes this study.

Preliminaries

Before proceeding further, let me first recall some fundamental concepts and properties in nano topological spaces.

Definition 2.1 [8] Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U. The pair (U,R) is said to be the approximation space. Let $X \subseteq U$:

1. The lower approximation of X with respect to R is denoted by $L_R(X) = \cup_{x \in U}\{R(x) : R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is denoted by $U_R(X) = \cup_{x \in U}\{R(x) : R(x) \cap X \neq \phi\}$.
3. The boundary region of X with respect to R is denoted by $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 [1] Let U be the universe, R be an equivalence relation on U and $X \subseteq U$. The nano topology on U with respect to X is defined by $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ and $(U, \tau_R(X))$ is called the nano topological space. The elements of $\tau_R(X)$ are called nano open sets and the complement of any nano open set is called nano closed set.

Definition 2.3 [1] Let $(U, \tau_R(X))$ be a nano topological space with respect to X, where $X \subseteq U$. For a subset $A \subseteq U$:

1. The nano interior of A is defined as the union of all nano open subsets contained in A and is denoted by $\mathrm{nint}(A)$.
2. The nano closure of A is defined as the intersection of all nano closed subsets containing A, and is denoted by $\mathrm{ncl}(A)$.

It should be noted that the concepts of the nano interior and the nano closure have the same topological characterizations and properties of the concepts of interior and the closure in the general topology.

Definition 2.4 [1, 2, 4] Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The set A is said to be:

1. Nano regular open, if $A = \mathrm{nint}(\mathrm{ncl}(A))$.
2. Nano α-open, if $A \subseteq \mathrm{nint}[\mathrm{ncl}(\mathrm{nint}(A))]$.
3. Nano semi-open, if $A \subseteq \mathrm{ncl}(\mathrm{nint}(A))$.
4. Nano pre-open, if $A \subseteq \mathrm{nint}(\mathrm{ncl}(A))$.
5. Nano γ-open, if $A \subseteq \mathrm{nint}(\mathrm{ncl}(A)) \cup \mathrm{ncl}(\mathrm{nint}(A))$.
6. Nano β-open (nano semi pre-open), if $A \subseteq \mathrm{ncl}[\mathrm{nint}(\mathrm{ncl}(A))]$.
7. Nano δ-closed, if $A = \mathrm{ncl}^{\delta}(A)$, where $\mathrm{ncl}^{\delta}(A) = \{x \in U : A \text{ and } x \in G\}$.
8. Nano $\delta\beta$-open, if $A \subseteq \mathrm{ncl}[\mathrm{nint}(\mathrm{ncl}^{\delta}(A))]$.

The family of all nano regular open (respectively, nano α-open, nano semi-open, nano pre-open, nano γ-open, nano β-open and nano $\delta\beta$-open) sets in a nano topological space $(U, \tau_R(X))$ is denoted by $NRO(U, X)$ (respectively, $N_\alpha(U, X)$, $NSO(U, X)$, $NPO(U, X)$, $N_\gamma(U, X)$, $N_\beta(U, X)$) and $N_{\delta\beta}O(U, X)$.

The relationships between nano near open sets are presented in Fig. 1 [4].

Fig. 1 The relationships between nano near open sets
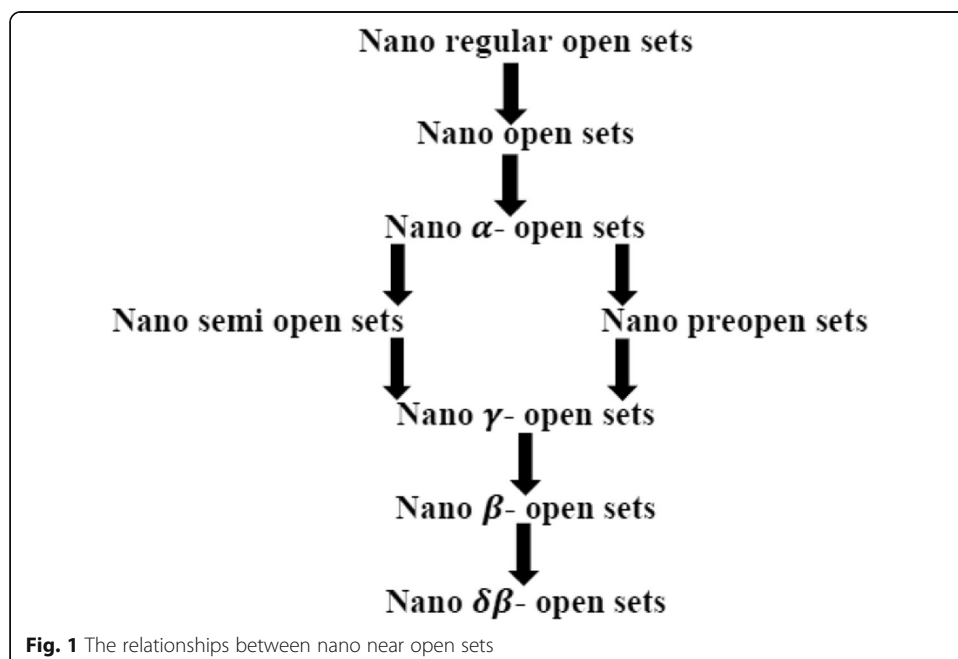
Definition 2.5 [1, 2, 4] A subset K of a nano topological space $(U, \tau_R(X))$ is called nano regular closed (respectively, nano α-closed, nano semi-closed, nano pre-closed, nano γ-closed, nano β-closed, and nano $\delta\beta$-closed) if its complement is nano regular open (respectively, nano α-open, nano semi-open, nano pre-open, nano γ-open, nano β-open and nano $\delta\beta$-open).

Definition 2.6 Let $(U, \tau_R(X))$ and $(V, \tau^*_{R'}(Y))$ be nano topological spaces. A mapping $f : (U, \tau_R(X)) \rightarrow (V, \tau^*_{R'}(Y))$ is said to be:

1. Nano continuous [5] if $f^{-1}(B)$ is nano open set in U for every nano open set B in V.
2. Nano α-continuous [6] if $f^{-1}(B)$ is nano α-open set in U for every nano open set B in V.
3. Nano semi-continuous [6] if $f^{-1}(B)$ is a nano semi-open set in U for every nano open set B in V.
4. Nano pre-continuous [6] if $f^{-1}(B)$ is a nano pre-open set in U for every nano open set B in V.
5. Nano γ-continuous [7] if $f^{-1}(B)$ is a nano γ-open set in U for every nano open set B in V.
6. Nano β-continuous [3] if $f^{-1}(B)$ is a nano β-open set in U for every nano open set B in V.
7. Nano $\delta\beta$-continuous [4] if $f^{-1}(B)$ is a nano $\delta\beta$-open set in U for every nano open set B in V.

In the paper [4] the relationships between the different types of near nano continuous functions are studied as shown in the following diagram in Fig. 2.

Throughout this paper $(U, \tau_R(X))$ is a nano topological space with respect to X where $X \subseteq U$, R is an equivalence relation on U, and U/R denotes the family of equivalence classes of U by R.

Fig. 2 The relationships between near nano continuous functions
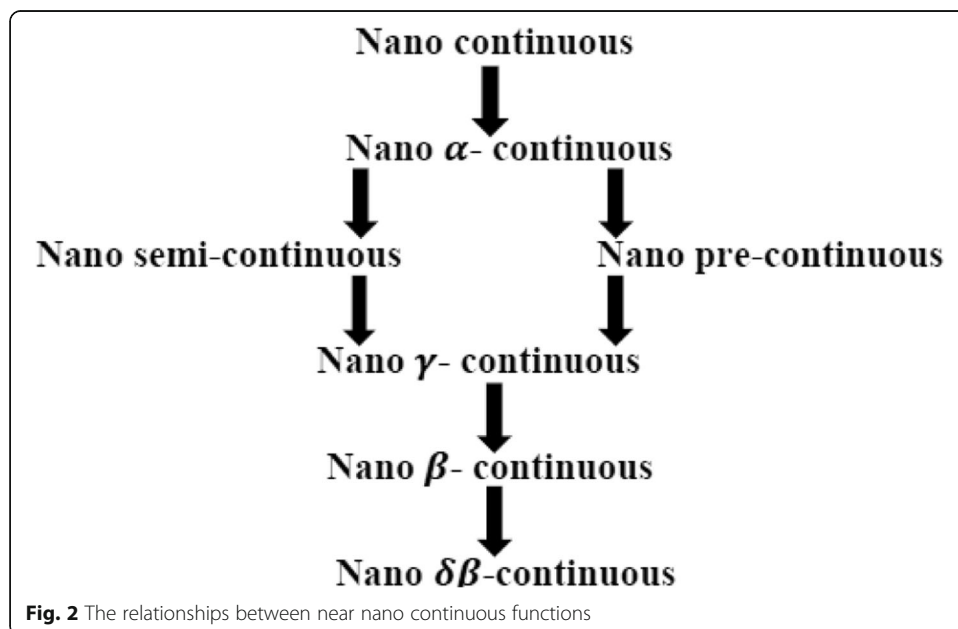
Nano $\wedge_\beta$-sets

This section presents a new notion in nano topology which is called nano $\wedge_\beta$-set. In addition, i indicates some nano topological properties of these sets. The results show that the proposed sets generalize the usual notions of nano near open sets [1, 2], whereas it is independent and different from nano $\delta\beta$-open sets [4].

Definition 3.1 Let $(U, \tau_R(X))$ be a nano topological space and A naon $\subseteq U$.

A nano subset $N - \wedge_\beta(A)$ is defined as follows: $N - \wedge_\beta(A) = \cap \{G : A \quad G, G \in N_\beta(U, X)\}$. The complement of $N - \wedge_\beta(A)$-set is called $N - \vee_\beta(A)$-set.

In the following lemma, I summarize the fundamental properties of the set $N - \wedge_\beta$.

Lemma 3.1 For subsets A, B and $A_\alpha (\alpha \in \Delta)$ of a nano topological space $(U, \tau_R(X))$, the following hold:

1. $A \subseteq N\text{-}\wedge_\beta(A)$.
2. If $A \subseteq B$, then $N\text{-}\wedge_\beta(A) \subseteq N - \wedge_\beta(B)$.
3. $N\text{-}\wedge_\beta(N\text{-}\wedge_\beta(A)) = N\text{-}\wedge_\beta(A)$.
4. If $A \in N_\beta(U, X))$, then $A = N\text{-}\wedge_\beta(A)$.
5. $N\text{-}\wedge_\beta(\cup \{A_\alpha : \alpha \in \Delta\}) = \cup \{N\text{-}\wedge_\beta(A_\alpha) : \alpha \in \Delta\}$.
6. $N\text{-}\wedge_\beta(\cap \{A_\alpha : \alpha \in \Delta\}) \subseteq \cap \{N\text{-}\wedge_\beta(A_\alpha) : \alpha \in \Delta\}$.

Proof. I prove only (5) and (6) since the others are consequences of Definition 3.1.

First, for each $\alpha \in \Delta$, $N\text{-}\wedge_\beta(A_\alpha) \subseteq N\text{-}\wedge_\beta(\cup_{\alpha \in \Delta} A_\alpha)$. Hence, $\cup_{\alpha \in \Delta} N\text{-}\wedge_\beta(A_\alpha) \subseteq N\text{-}\wedge_\beta(\cup_{\alpha \in \Delta} A_\alpha)$. Conversely, suppose that $x \notin \cup_{\alpha \in \Delta} N\text{-}\wedge_\beta(A_\alpha)$. Then, $x \notin N\text{-}\wedge_\beta(A_\alpha)$ for each $\alpha \in \Delta$ and hence there exists $G_\alpha \in N_\beta(U, X)$ such that $A_\alpha \subseteq G_\alpha$ and $x \notin G_\alpha$ for each $\alpha \in \Delta$. We have that $\cup_{\alpha \in \Delta} A_\alpha \subseteq \cup_{\alpha \in \Delta} G_\alpha$ and $\cup_{\alpha \in \Delta} G_\alpha$ is a nano β-open set which does not contain x. Therefore, $x \notin N\text{-}\wedge_\beta(\cup_{\alpha \in \Delta} A_\alpha)$. Thus, $N\text{-}\wedge_\beta(\cup_{\alpha \in \Delta} A_\alpha) \subseteq \cup_{\alpha \in \Delta} N\text{-}\wedge_\beta(A_\alpha)$.

Suppose that, $x \notin \cap \{N - \wedge_\beta(A_\alpha) : \alpha \in \Delta\}$. There exists $\alpha_0 \in \Delta$ such that $x \notin N - \wedge_\beta(A_{\alpha_0})$, and there exists a nano β-open set G such that $x \notin G$ and $A_{\alpha_0} \subseteq G$. We have that $\cap_{\alpha \in \Delta} A_\alpha \subseteq A_{\alpha_0} \subseteq G$ and $x \notin G$. Therefore, $x \notin N - \wedge_\beta(\cap \{A_\alpha : \alpha \in \Delta\})$.

Remark 3.1 The inclusion in Lemma 3.1 parts 1 and 6 cannot be replaced by equality relation. Moreover, the converse of part 2 is not necessarily true as shown in the following example.

Example 3.1 Let $U = \{a, b, c, d\}$ with $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{c\}$. Then, $\tau_R(X) = \{U, \phi, \{c\}\}$.

i.   For part 1, if $A = \{a\}$, then $N - \wedge_\beta(A) = \{a, c\}$, and $N - \wedge_\beta(A) \nsubseteq A$.
ii. For part 6, if $A = \{a\}$, and $B = \{b\}$, then $A \cap B = \phi$, and $N - \wedge_\beta(A) = \{a, c\}$, $N - \wedge_\beta(B) = \{b, c\}$, $N - \wedge_\beta(A \cap B) = \phi$, and $N - \wedge_\beta(A) \cap N - \wedge_\beta(B) = \{c\} \nsubseteq N - \wedge_\beta(A \cap B) = \phi$.
iii. For part 2, if $A = \{c\}$, and $B = \{a\}$, then $N - \wedge_\beta(A) = A$, and $N - \wedge_\beta(B) = \{a, c\}$. Therefore, $N - \wedge_\beta(A) \subseteq N - \wedge_\beta(B)$, but $A \nsubseteq B$.

Definition 3.2 Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. A subset A is called nano $\wedge_\beta$-set if $A = N - \wedge_\beta(A)$. The complement of nano $\wedge_\beta$-set is nano $\vee_\beta$-set. The family of all nano $\wedge_\beta$-sets and nano $\vee_\beta$-sets are denoted by $N - \tau^{\wedge_\beta}$ and $N - \Gamma^{\vee_\beta}$, respectively.

In Example 3.1, $N - \tau^{\wedge_\beta} = \{U, \phi, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and $N - \Gamma^{\vee_\beta} = \{U, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$.

In the following lemma, i summarize the fundamental properties of nano $\wedge_\beta$-sets which show that nano $\wedge_\beta$-sets are a generalization of nano β-open sets [2].

Lemma 3.2 For subsets A, B and $A_\alpha (\alpha \in \Delta)$ of a nano topological space $(U, \tau_R(X))$, the following properties hold:

1. $N - \wedge_\beta(A)$, U, ϕ are nano $\wedge_\beta$-sets.
2. If A is a nano-β-open set, then A is a nano $\wedge_\beta$-set.
3. If A_α is a nano $\wedge_\beta$-set $\forall \alpha \in \Delta$, then $\cup_{\alpha \in \Delta} A_\alpha$ is a nano $\wedge_\beta$-set.
4. If A_α is a nano $\wedge_\beta$-set $\forall \alpha \in \Delta$, then $\cap_{\alpha \in \Delta} A_\alpha$ is a nano $\wedge_\beta$-set.

Proof. This follows from Lemma 3.1.

Remark 3.2 It is clear from (1), (3), and (4) in Lemma 3.2 that the family of all nano $\wedge_\beta$-sets forms a topology.
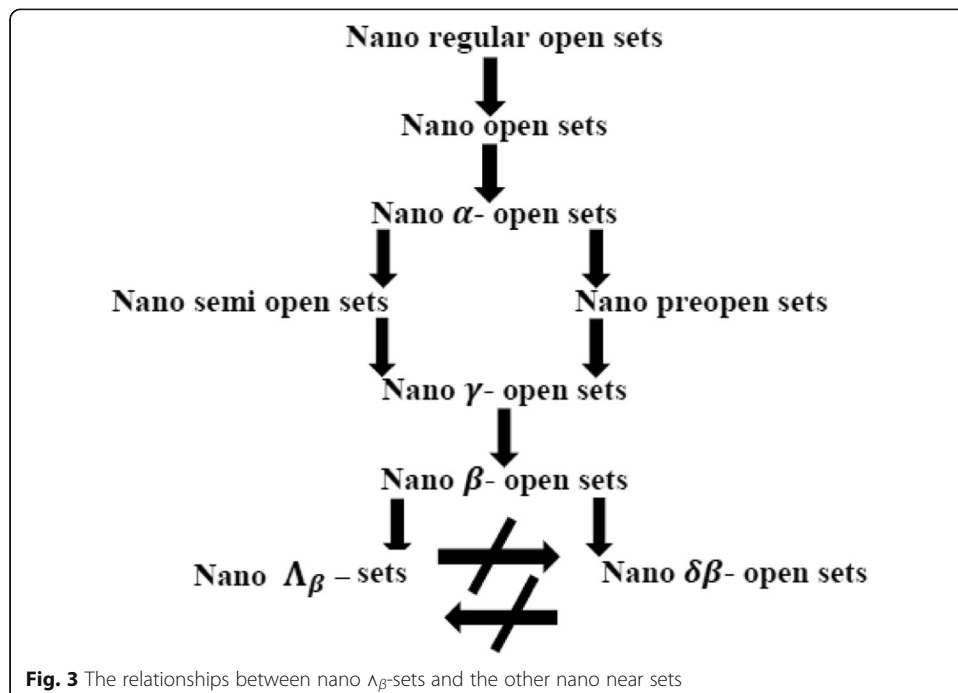
In (2) in Lemma 3.2, the converse is not necessarily true as shown in the following example.

Example 3.2 Let $U = \{a, b, c, d\}$, $U/R = \{\{a, b\}, \{c\}, \{d\}\}$, and $X = \{a, c\}$. Then, $\tau_R(X) = \{U, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$. If $A = \{d\}$, then A is a nano $\wedge_\beta$-set, but $A = \{d\}$ is not a nano β-open set.

Remark 3.3 The nano $\delta\beta$-open sets of Definition 2.4 [4] and the current Definition 3.2 of nano $\wedge_\beta$-sets are different and independent as shown in Fig. 3. Example 3.1 shows that $\{a\}$ is a nano $\delta\beta$-open set, but it is not a nano $\wedge_\beta$-set. Moreover, Example 3.2 shows that $\{d\}$ is a nano $\wedge_\beta$-set, but it is not a nano $\delta\beta$-open set.

Corollary 3.1 Let $(U, \tau_R(X))$ be a nano topological space. Then,

1. $\mathrm{NSO}(U, X) \cup \mathrm{NPO}(U, X) \subseteq N - \tau^{\wedge_\beta}$.
2. $\mathrm{NSO}(U, X) \cap \mathrm{NPO}(U, X) \subseteq N - \tau^{\wedge_\beta}$.

Fig. 3 The relationships between nano $\wedge_\beta$-sets and the other nano near sets

Remark 3.4 *The equality in Corollary 3.1 does not hold in general. In Example 3.2, the set A = {d} is nano $\wedge_\beta$-set but it is neither in NSO(U, X) ∪ NPO(U, X) nor in NSO(U, X) ∩ NPO(U, X).*

Proposition 3.1 *The intersection of a nano open and a nano $\wedge_\beta$-set is a nano $\wedge_\beta$-set.*

Proof. Let A be a nano open and B be a nano $\wedge_\beta$-open. Then, $A = N\text{-}\wedge_\beta(A)$ and $B = N\text{-}\wedge_\beta(B)$.

$$
\begin{aligned}
A \cap B &= N{-}\wedge_\beta(A) \cap N{-}\wedge_\beta(B), \\
&\supseteq N{-}\wedge_\beta(A \cap B).
\end{aligned}
\tag{1}
$$

Therefore, $N - \wedge_\beta(A \cap B) \subseteq (A \cap B)$, but $N - \wedge_\beta(A \cap B) \supseteq (A \cap B)$ from Lemma 3.1 (1).

Proposition 3.2 *If A and B are nano subsets of U such that $A \subseteq B \subseteq \mathrm{ncl}(\mathrm{nint}(A))$, then B is a nano $\wedge_\beta$-set in U.*

Proof. It clear from Definition 2.3 [1] that,

$$
\begin{aligned}
\mathrm{nint}(A) &\subseteq \mathrm{ncl}(A), \\
\Rightarrow \mathrm{nint}(\mathrm{nint}(A)) &\subseteq \mathrm{nint}(\mathrm{ncl}(A)), \\
\Rightarrow \mathrm{nint}(A) &\subseteq \mathrm{nint}(\mathrm{ncl}(A)), \\
\Rightarrow \mathrm{ncl}(\mathrm{nint}(A)) &\subseteq \mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))).
\end{aligned}
\tag{2}
$$

Then,

$$
\begin{aligned}
B &\subseteq \mathrm{ncl}(\mathrm{nint}(A)), \\
&\subseteq \mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))), \\
&\subseteq \mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(B))).
\end{aligned}
\tag{3}
$$

Hence, $B \subseteq \mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(B)))$. Therefore B is nano β-open in U. Thus, by Lemma 3.2 (2) B is a nano $\wedge_\beta$-set in U.

Definition 3.3 *Let $(U, \tau_R(X))$ be a nano topological space and $A \subseteq U$. The nano $\vee_\beta$-closure of a set A, denoted by $\mathrm{ncl}^{\vee_\beta}(A)$, is the intersection of nano $\vee_\beta$-sets including A. The nano $\wedge_\beta$-interior of a set A, denoted by $\mathrm{nint}^{\wedge_\beta}(A)$, is the union of nano $\wedge_\beta$-sets included in A.*

The following theorem presents the main properties of nano $\vee_\beta$-closure and nano $\wedge_\beta$-interior which are required in the sequel to study the properties of nano $\wedge_\beta$-continuous functions.

Theorem 3.1 *Let $(U, \tau_R(X))$ be a nano topological space and $A, B \subseteq U$. Then, the following properties hold:*

1. $\mathrm{ncl}^{\vee_\beta}(A)$ is a nano $\vee_\beta$-set and $\mathrm{nint}^{\wedge_\beta}(A)$ is a nano $\wedge_\beta$-set.
2. $\mathrm{nint}^{\wedge_\beta}(A) \subseteq A \subseteq \mathrm{ncl}^{\vee_\beta}(A)$.
3. $A = \mathrm{ncl}^{\vee_\beta}(A)$ iff A is a nano $\vee_\beta$-set and $\mathrm{nint}^{\wedge_\beta}(A) = A$ iff A is a nano $\wedge_\beta$-set.
4. If $A \subseteq B$, then $\mathrm{ncl}^{\vee_\beta}(A) \subseteq \mathrm{ncl}^{\vee_\beta}(B)$ and $\mathrm{nint}^{\wedge_\beta}(A) \subseteq \mathrm{nint}^{\wedge_\beta}(B)$.
5. $\mathrm{nint}^{\wedge_\beta}(A) \cup \mathrm{nint}^{\wedge_\beta}(B) \subseteq \mathrm{nint}^{\wedge_\beta}(A \cup B)$.
6. $\mathrm{ncl}^{\vee_\beta}(A \cap B) \subseteq \mathrm{ncl}^{\vee_\beta}(A) \cap \mathrm{ncl}^{\vee_\beta}(B)$.

Proof. Obvious.

Remark 3.5 *Example 3.1 shows that*

1. The inclusion in Theorem 3.1 parts 2, 5, and 6 can not be replaced by equality relation:

    i.   For part 2, if $A = \{a, b, c\}$, then $\mathrm{ncl}^{\vee_\beta}(A) = U$ and so, $\mathrm{ncl}^{\vee_\beta}(A) \not\subseteq A$. If $A = \{a\}$, then $\mathrm{nint}^{\wedge_\beta}(A) = \phi$ and thus, $\mathrm{nint}^{\wedge_\beta}(A) \not\subseteq A$.

    ii. For part 5, if $A = \{a\}$ and $B = \{c\}$, then $A \cup B = \{a, c\}$, $\mathrm{nint}^{\wedge_\beta}(A) = \phi$, $\mathrm{nint}^{\wedge_\beta}(B) = \{c\}$, $\mathrm{nint}^{\wedge_\beta}(A \cup B) = A \cup B$ and so, $\mathrm{nint}^{\wedge_\beta}(A \cup B) = \{a, c\} \not\subseteq \{c\} = \mathrm{nint}^{\wedge_\beta}(A) \cup \mathrm{nint}^{\wedge_\beta}(B)$.

    iii. For part 6, if $A = \{d\}$ and $B = \{a, b, c\}$, then $A \cap B = \phi$, $\mathrm{ncl}^{\vee_\beta}(A) = A$, $\mathrm{ncl}^{\vee_\beta}(B) = U$, $\mathrm{ncl}^{\wedge_\beta}(A \cap B) = \phi$ and so, $\mathrm{ncl}^{\vee_\beta}(A) \cap \mathrm{ncl}^{\vee_\beta}(B) = \{d\} \not\subseteq \mathrm{ncl}^{\vee_\beta}(A \cap B) = \phi$.

2. The converse of part 4 is not necessarily true:

    i.   If $A = \{d\}$ and $B = \{a, b, c\}$, then $\mathrm{ncl}^{\vee_\beta}(A) = A$ and $\mathrm{ncl}^{\vee_\beta}(B) = U$. Therefore, $\mathrm{ncl}^{\vee_\beta}(A) \subseteq \mathrm{ncl}^{\vee_\beta}(B)$, but $A \not\subseteq B$.

    ii. If $A = \{a\}$ and $B = \{c\}$, then $\mathrm{nint}^{\wedge_\beta}(A) = \phi$ and $\mathrm{nint}^{\wedge_\beta}(B) = B$. Therefore, $\mathrm{nint}^{\wedge_\beta}(A) \subseteq \mathrm{nint}^{\wedge_\beta}(B)$, but $A \not\subseteq B$.

Nano $\wedge_\beta$-sets and lower and upper approximations

The goal of this section is to investigate various forms of nano $\wedge_\beta$-sets corresponding to different cases of approximations.

Proposition 4.1 *If $U_R(X) = U$ in a nano topological space, then N-$\tau^{\wedge_\beta}$ is $P(U)$.*

Proof. Let $U_R(X) = U$.

1. If $L_R(X) = \phi$, then $B_R(X) = U$ and $\tau_R(X) = \{U, \phi\}$. Hence, $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = U \, \forall \, A \subseteq U$, $A \neq \phi$. So $A \subseteq \mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) \, \forall \, A \subseteq U$, $A \neq \phi$. Therefore, A is nano β-open in U. Thus, $N_\beta(U, X)$ is $P(U)$. Consequently, $N - \tau^{\wedge_\beta}$ is $P(U)$.

2. If $L_R(X) \neq \phi$, then $\tau_R(X) = \{U, \phi, L_R(X), [L_R(X)]'\}$.

    i.   If $A \subseteq L_R(X)$, then $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = L_R(X)$. Therefore, A is nano β-open in U. Consequently, N-$\tau^{\wedge_\beta}$ is $P(U)$.

    ii. If $A \subseteq [L_R(X)]'$, then $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = [L_R(X)]'$. Therefore, A is nano β-open in U. Consequently, N-$\tau^{\wedge_\beta}$ is $P(U)$.

    iii. If $A \cap L_R(X) \neq \phi$ and $A \not\subseteq L_R(X)$, then $\mathrm{ncl}(A) = U$ and $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = U$. Therefore, A is nano β-open in U. Thus, $N_\beta(U, X)$ is $P(U)$. Consequently, N-$\tau^{\wedge_\beta}$ is $P(U)$.

    iv. If $A \cap [L_R(X)]' \neq \phi$ and $A \not\subseteq [L_R(X)]'$, then $\mathrm{ncl}(A) = U$ and $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = U$. Therefore, A is nano β-open in U. Thus, $N_\beta(U, X)$ is $P(U)$. Consequently, N-$\tau^{\wedge_\beta}$ is $P(U)$.

Proposition 4.2 *If $U_R(X) \neq U$ in a nano topological space, then U, ϕ and any sets which intersects $U_R(X)$ are nano $\wedge_\beta$-sets in U.*

Proof. Let $U_R(X) \neq U$.

1. If $(L_R(X) = \phi$ or $L_R(X) = U_R(X))$. In both cases, $\tau_R(X) = \{U, \phi, U_R(X)\}$. If A intersects $U_R(X)$, then $\mathrm{ncl}(A) = U$, hence $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = U$. Therefore, A is a nano β-open set in U.

2. If $L_R(X) \neq \phi$, then $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$. Let A intersect $U_R(X)$.

    i.   If $A \cap L_R(X) \neq \phi$, then $\mathrm{ncl}(A) = [B_R(X)]'$, $\mathrm{nint}(\mathrm{ncl}(A)) = L_R(X)$, $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = [B_R(X)]' = [U_R(X)]' \cup L_R(X) \supseteq A$. Therefore, A is nano β-open in U. Hence, it is a nano $\wedge_\beta$-set.

    ii. If $A \cap B_R(X) \neq \phi$, then $\mathrm{ncl}(A) = [L_R(X)]'$, $\mathrm{nint}(\mathrm{ncl}(A)) = B_R(X)$, $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = [L_R(X)]' = [U_R(X)]' \cup B_R(X) \supseteq A$. Therefore, A is nano β-open in U. Hence, it is a nano $\wedge_\beta$-set.

    iii. If $A \cap L_R(X) \neq \phi$ and $A \cap B_R(X) \neq \phi$, then $\mathrm{ncl}(A) = U$ hence $\mathrm{ncl}(\mathrm{nint}(\mathrm{ncl}(A))) = U$. Therefore, A is nano β-open in U. Hence, it is a nano $\wedge_\beta$-set.

Nano $\wedge_\beta$-continuity

The concepts of nano near continuous functions are extended to nano $\wedge_\beta$-continuous functions. It is shown that every nano β-continuous function is nano $\wedge_\beta$-continuous function. Therefore, the nano $\wedge_\beta$-continuous functions are generalization of the other types of nano near continuous functions [3, 5–7].

Definition 5.1 *Let $(U, \tau_R(X))$ and $(V, \tau_{R'}^*(Y))$ be nano topological spaces. A mapping $f : (U, \tau_R(X)) \rightarrow (V, \tau_{R'}^*(Y))$ is said to be a nano $\wedge_\beta$-continuous function if $f^{-1}(B)$ is a nano $\wedge_\beta$-set in U for every nano open set B in V.*

The relationships between nano β-continuous and nano $\wedge_\beta$-continuous functions are given in the following remark.

Remark 5.1 *Every nano β-continuous is nano $\wedge_\beta$-continuous.*

The converse of Remark 5.1 is not necessarily true as shown in the following example.

Example 5.1 *Let $U = \{a, b, c, d\}$ with $U/R = \{\{b, c\}, \{a\}, \{d\}\}$ and $X = \{a, d\}$. Then, $\tau_R(X) = \{U, \phi, \{a, d\}\}$. Let $V = \{x, y, z, w\}$ with $V/R' = \{\{z\}, \{w\}, \{x, y\}\}$, $Y = \{x, w\}$. Then, $\tau_R^*(Y) = \{V, \phi, \{w\}, \{x, y\}, \{x, y, w\}\}$.*

Define $f : U \rightarrow V$ as $f(a) = f(d) = x$, $f(c) = z$, $f(b) = w$. Then f is a nano $\wedge_\beta$-continuous, but it is not nano β-continuous.

Remark 5.2 *The nano $\delta\beta$-continuous of Definition 2.6 [4] and the current Definition 5.1 of nano $\wedge_\beta$-continuous are different and independent as shown in Fig. 4. In Example 5.1, let $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{a, c\}$. Then, $\tau_R(X) = \{U, \phi, \{c\}, \{a, b\}, \{a, b, c\}\}$.*

Define $f : U \rightarrow V$ as $f(a) = f(b) = x$, $f(c) = z$, $f(d) = w$, then f is a nano $\wedge_\beta$-continuous, but it is not nano $\delta\beta$-continuous. Moreover, let $U/R = \{\{a, b\}, \{c\}, \{d\}\}$ and $X = \{c\}$ in Example 5.1. Then, f is nano $\delta\beta$-continuous, but it is not nano $\wedge_\beta$-continuous.

It should be noted that from Remarks 5.1 and 5.2 and Fig. 2, I present Fig. 4 which shows that the current Definition 5.1 is a generalization of Definition 2.6 in [3, 5–7]. Additionally, it is different and independent of Definition 2.6 [4].

Fig. 4 The relationships between the different types of nano near continuous functions
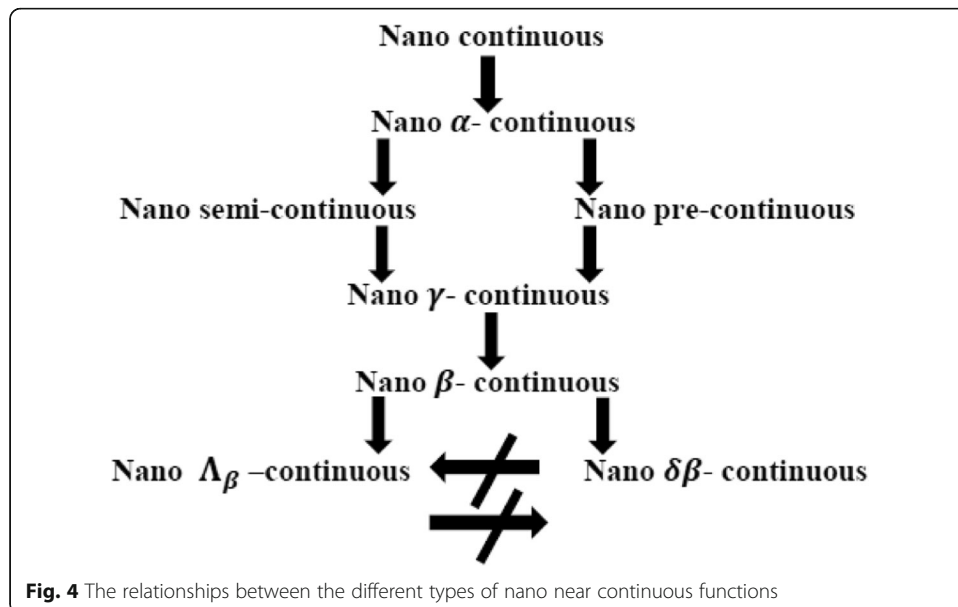
Theorem 5.1 *Let $(U, \tau_R(X))$ and $(V, \tau_{R'}^*(Y))$ be nano topological spaces and let $f : (U, \tau_R(X)) \to (V, \tau_{R'}^*(Y))$ be a mapping. Then, the following statements are equivalent:*

1. *f is nano $\wedge_\beta$-continuous.*
2. *The inverse image of every nano closed set G in V is nano $\vee_\beta$-set in U.*
3. *$f(\text{ncl}^{\vee_\beta}(A)) \subseteq \text{ncl}(f(A)), \forall A \subseteq U$.*
4. *$\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq f^{-1}(\text{ncl}(F)), \forall F \subseteq V$.*
5. *$f^{-1}(\text{nint}(F)) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(F)), \forall F \subseteq V$.*

Proof.

1. $(1) \Rightarrow (2)$; let f be nano $\wedge_\beta$-continuous and let G be a nano closed set in V. Then, $V - G$ is nano open in V. Since, f is nano $\wedge_\beta$-continuous. Then, $f^{-1}(V - G)$ is a nano $\wedge_\beta$-set in U. Then, $f^{-1}(V - G) = U - f^{-1}(G)$ and hence $f^{-1}(G)$ is nano $\vee_\beta$-set in U.
2. $(2) \Rightarrow (1)$; let A be a nano open set in V. Then, $f^{-1}(V - A)$ is nano $\vee_\beta$-set in U. Then, $f^{-1}(A)$ is nano $\wedge_\beta$-set in U. Therefore, f is nano $\wedge_\beta$-continuous.
3. $(1) \Rightarrow (3)$; let f be nano $\wedge_\beta$-continuous and let $A \subseteq U$. Since, f is nano $\wedge_\beta$-continuous and $\text{ncl}(f(A))$ is nano closed in V, $f^{-1}(\text{ncl}(f(A)))$ is a nano $\vee_\beta$-set in U. Since, $f(A) \subseteq \text{ncl}(f(A))$, $f^{-1}(f(A)) \subseteq f^{-1}(\text{ncl}(f(A)))$, then $\text{ncl}^{\vee_\beta}(A) \subseteq \text{ncl}^{\vee_\beta}(f^{-1}(\text{ncl}(f(A))))$. Thus, $\text{ncl}^{\vee_\beta}(A) \subseteq f^{-1}(\text{ncl}(f(A)))$. Therefore, $f(\text{ncl}^{\vee_\beta}(A)) \subseteq \text{ncl}(f(A)), \forall A \subset U$.
4. $(3) \Rightarrow (1)$; let $f(\text{ncl}^{\vee_\beta}(A)) \subseteq \text{ncl}(f(A)), \forall A \subseteq U$ and let F be nano closed in V. Then, $f^{-1}(F) \subseteq U$. Thus, $f(\text{ncl}^{\vee_\beta}(f^{-1}(F))) \subseteq \text{ncl}(f(f^{-1}(F))) \subseteq \text{ncl}(F) = F$ that is $\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus, $\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq f^{-1}(F)$, but $f^{-1}(F) \subseteq \text{ncl}^{\vee_\beta}(f^{-1}(F))$. Hence, $\text{ncl}^{\vee_\beta}(f^{-1}(F)) = f^{-1}(F)$. Therefore, $f^{-1}(F)$ is nano $\vee_\beta$-closed in U and hence f is nano $\wedge_\beta$-continuous.
5. $(1) \Rightarrow (4)$; let f be nano $\wedge_\beta$-continuous and let $F \subseteq V$. Since, $F \subseteq \text{ncl}(F)$, then $f^{-1}(F) \subseteq f^{-1}(\text{ncl}(F))$ and hence $\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq \text{ncl}^{\vee_\beta}(f^{-1}(\text{ncl}(F))) = f^{-1}(\text{ncl}(F))$ as $\text{ncl}(F)$ is nano closed in V and f is nano $\wedge_\beta$-continuous. Thus, $\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq f^{-1}(\text{ncl}(F))$.
6. $(4) \Rightarrow (1)$; let $\text{ncl}^{\vee_\beta}(f^{-1}(F)) \subseteq f^{-1}(\text{ncl}(F)), \forall F \subseteq V$ and let G be nano closed in V. Then, $\text{ncl}^{\vee_\beta}(f^{-1}(G)) \subseteq f^{-1}(\text{ncl}(G)) = f^{-1}(G)$. Hence, $\text{ncl}^{\vee_\beta}(f^{-1}(G)) \subseteq f^{-1}(G)$, but $f^{-1}(G) \subseteq \text{ncl}^{\vee_\beta}(f^{-1}(G))$. Therefore, $\text{ncl}^{\vee_\beta}(f^{-1}(G)) = f^{-1}(G)$ and hence f is nano $\wedge_\beta$-continuous.
7. $(1) \Rightarrow (5)$; let f be nano $\wedge_\beta$-continuous and let $F \subseteq V$. Since, $\text{nint}(F)$ is nano open in V, then $f^{-1}(\text{nint}(F))$ is a nano $\wedge_\beta$-set in U. Therefore, $\text{nint}^{\wedge_\beta}[f^{-1}(\text{nint}(F))] = f^{-1}(\text{nint}(F))$. Also, $\text{nint}(F) \subseteq F$ implies that $f^{-1}(\text{nint}(F)) \subseteq f^{-1}(F)$. Therefore, $\text{nint}^{\wedge_\beta}(f^{-1}(\text{nint}(F))) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(F))$. That is, $f^{-1}(\text{nint}(F)) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(F))$.
8. $(5) \Rightarrow (1)$; let $f^{-1}(\text{nint}(F)) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(F))$, for every subset F of V and let G be nano open in V. Then $\text{nint}(G) = G$. By assumption, $f^{-1}(\text{nint}(G)) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(G))$. Thus, $f^{-1}(G) \subseteq \text{nint}^{\wedge_\beta}(f^{-1}(G))$. But $\text{nint}^{\wedge_\beta}(f^{-1}(G)) \subseteq f^{-1}(G)$. Therefore, $f^{-1}(G) = \text{nint}^{\wedge_\beta}(f^{-1}(G))$ and hence f is nano $\wedge_\beta$-continuous.

Remark 5.3 *In Theorem 5.1 the equality of parts 3, 4, and 5 does not hold in general as shown in Example 5.1 that:*

1. for part 3, take $A = \{b\}$; then, $f(\text{ncl}^{\vee_\beta}(A)) = \{w\} \neq \text{ncl}(f(A)) = V$.
2. for part 4, take $F = \{w\}$; then, $\text{ncl}^{\vee_\beta}(f^{-1}(F)) = \{b\} \neq f^{-1}(\text{ncl}(F)) = U$.

3. for part 5, take $F = \{x\}$; then, $f^{-1}(\mathrm{nint}(F)) = \phi \neq \mathrm{nint}^{\wedge_\beta}(f^{-1}(F)) = \{a, d\}$.

Conclusions

I have endeavored to generalize and extend the several weaker forms of nano open sets. In this paper, a new expansion through the nano topological structure was studied and analyzed. Some important characteristics and main properties which were related to these sets were obtained. Eventually, i depict the relationships among the weaker forms of nano open sets and nano $\wedge_\beta$-sets. The results showed that the proposed sets generalized the nano near open sets [1, 2], whereas this new notion is independent and different from nano $\delta\beta$-open sets [4]. The concepts of nano near continuous functions were extended to nano $\wedge_\beta$-continuous functions and the corresponding properties were examined. It was shown that the nano $\wedge_\beta$-continuous functions were a generalization of the other types of nano near continuous functions [3, 5–7] and they were different from nano $\delta\beta$-continuous functions [4].

References
1. Thivagar, M.L., Richard, C.: On nano forms of weakly open sets. Int. J. Math. Stat. Inven. **1**(1), 31–37 (2013)
2. Revathy, A., Ilango, G.: On nano β -open sets. Int. J. Eng. Contemp. Math. Sci. **1**(2), 1–6 (2015)
3. Nasef, A.A., Aggour, A.I., Darwesh, S.M.: On some classes of nearly open sets in nano topological spaces. J. Egypt. Math. Soc. **24**(4), 585–589 (2016)
4. M. Hosny, Nano $\delta\beta$ -open sets and nano $\delta\beta$-continuity. J. Egypt. Math. Soc. **26**(2), 365–375 (2018)
5. Thivagar, M.L., Richard, C.: On nano continuity. Math. Theory Model. **7**, 32–37 (2013)
6. Mary, D.A., Arockiarani, I.: On characterizations of nano *rgb*-clased sets in nano topological spaces. Int. J. Mod. Eng. Res. **5**(1), 68–76 (2015)
7. Mary, D.A., Arockiarani, I.: On *b*-open sets and *b*-continuous functions in nano topological spaces. Int. J. Innov. Res. Stud. **3**(11), 98–116 (2014)
8. Pawlak, Z.: Rough sets. Int. J. Comput. Inform. Sci. **11**(5), 341–356 (1982)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.